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Dr. H. Knox-Shaw, President, in the Chair.

The election by the Council of the following Fellows was duly confirmed :—

Loring B. Andrews, Astronomical Laboratory, Harvard University, Cambridge, Mass., U.S.A. ;

Mrs. Violet Olivia Cressy-Marcks, Broad Walk, Buxton, Derbyshire, and 42 Park Lane, London, W. 1 ;

John Theodore Foxell, M.A., 12 Tavistock Road, Croydon, Surrey ;

H. Groot, 52 Huizerweg, Bussum, Holland ;

Miss Mira Francisca Hardcastle, Stephen's Orchard, Grayshott, Hindhead, Surrey ;

Charles Henry Kebby, 52 Lumley Road, Horley, Surrey ;

Alfred Knight, 595 Madison Avenue, New York City, N.Y., U.S.A. ;

Ernest Vivian Leembruggen, 31 Gordon Street, Suva, Fiji ;

Arumugam Wisuvalingam Mailvaganam, B.A., B.Sc., Emmanuel College, Cambridge ;

Edward B. Maitland, O.B.E., B.Sc., A.M.I.Mech.E., Craig House, Woodlands Road, Surbiton ;

Mrs. Marion Field Skinner, 17 Solle-shott West, Letchworth, Herts ;

Michael Heriot Huth Walters, B.A., King's College, Cambridge ;

Reginald Hugh Wickramasinghe, B.A., B.Sc., c/o The Colonial Secretary, Ceylon ;

Charles Llewelyn Carrick Williams, Beckets, Rusper, Sussex ; and

Kenneth Young, D.S.O., 40 Danethorpe Road, Wembley, Middlesex.

Fifty-six presents were announced as having been received since the last meeting, including :—

The Institution of Electrical Engineers, Faraday Centenary Celebrations, 1931.

Observatoire Royal de Belgique, Carte du Ciel, zones $+33°$, $+35°$, $+37°$, $+39°$ (19 charts).

The President announced that the Gold Medal of the Society had been awarded by the Council to Dr. Robert Grant Aitken, Director of the Lick Observatory, for his work on double stars.

IONIZATION IN STELLAR ATMOSPHERES.—PART III.*

E. A. Milne and S. Chandrasekhar.

1. *Introduction.*—Saha's explanation of the main features of stellar spectra in terms of high-temperature ionization involved the consideration of (*a*) the *concentration* of atoms of a given species in given states of ionization and excitation, (*b*) the *mean pressure* in the stellar atmosphere. The same considerations were also involved in the Fowler-Milne treatment of the " maxima " of absorption lines. But it is clear that the ideas of a " mean concentration " and a " mean-pressure " in the " stellar atmosphere " are all much too vague. An approach to a rational treatment of stellar spectra may be attempted by the fusion into the general theory of a useful idea due to Stewart and Unsöld, namely, the conception of the " number of atoms above the photosphere." Also, instead of the mean pressure, a much more satisfactory parameter to work with is the *actual* pressure at a definite layer. The theory of ionization on these more general lines has already been developed by Milne in Part I. This resulted in the introduction into the theory of what may be called " integrated ionization equations." In the maximum problems and in the discussion of the absolute magnitude effects consequent on this development in the ionization theory, the evaluation of the optical thickness τ as a function of the actual pressure (P_0) at a definite layer and the ionization constants (K_1, K_2, etc.) plays an important rôle. The procedure adopted in Parts I and II was to use a semi-empirical formula for the absorption coefficient κ, and then derive the validity or otherwise of the assumption from the predictions of the theory. Thus in Parts I and II the complete set of formulæ on the two assumptions (i) $\kappa =$ constant, $\bar{\kappa}$, and (ii) $\kappa \propto (P/(kT)^{\frac{5}{2}})$ were worked out, and from the predicted absolute magnitude

* This paper is a sequel to two earlier papers by Milne on the same subject : " Ionization in Stellar Atmospheres—Part I," *M.N.*, **89**, 17–49, 1928. " Ionization in Stellar Atmospheres—Part II," *ibid.*, **89**, 157–175, 1928.

effects consequent on these two assumptions Milne * and also Miss Payne † have concluded that the observational evidence *supports* $\kappa \propto P$ and definitely contradicts the assumption of constant κ. But this, of course, does not mean that we have here a detailed verification of the law of absorption. Indeed, as has been emphasized by Milne in Part II and in Bakerian Lecture, what is necessary to explain the observed absolute magnitude effects—say, the strengthening of the lines of the neutral atom as we pass from dwarfs to giants of the same colour type—is the conjoint working of two opposing tendencies. Thus, for a *given* optical thickness τ, a reduction of g will tend to cause the mass per unit area to increase, and this should more than counterbalance the decrease in the proportion of the neutral atoms due to the increase in the ionization which results from the decrease in pressure consequent on the decrease in g. Such a counterbalancing is necessary in order to have a positive absolute magnitude effect ($dN_0/dg < 0$) for the lines of the neutral atom. The former requirement, namely, a sufficient increase in the " mass of the material above the photosphere," is provided for by the law $\kappa \propto P$ but not by $\kappa =$ constant. But it does not follow that $\kappa \propto P$ is the only conceivable law that will satisfy this requirement. What the previous analysis does prove, however, is the impossibility of the law $\kappa =$ constant. Now, physical considerations show that κ is not merely a function of P, but depends also on the x_r's, the fractions of the atoms in the successive stages of ionization,‡ and since in the evaluation of N_0, N_1, N_2, etc., the variation of the x's in the different layers is accurately taken into account, it would follow that an accurate theory, in order to be consistent, should also provide for the dependence of the absorption coefficient on the x's when evaluating the optical thickness τ. Thus, if we consider a one-constituent atmosphere where the first-stage ionization is setting in, it is physically clear that $\kappa \propto Px^+$, where x^+ is the fraction of the atoms ionized, for the circumstance that a certain fraction $(1 - x^+)$ of the atoms retain their optical electrons for all time is precisely equivalent to the circumstance that just this fraction $(1 - x^+)$ of the total number of atoms does not take part in the " capture process," and hence is ineffective in the photo-electric absorption ; also, since these atoms are electrically neutral they cannot take part in the " free-free ' absorption either. Now if we use the law $\kappa \propto Px^+$, allowing accurately for the variation in x^+ down the column, we find immediately that (i) there is *identically* no absolute magnitude effect for the lines of the neutral atom, (ii) there is *identically* no " null effect " for the lines of the ionized atom, and (iii) the temperature at which dN_0/dT vanishes for constant τ is independent of P_0. We will not discuss the significance of these results at this stage, but this example suffices to show that a completion of the analysis started in Parts I and II, based on an accurate theory of the absorption coefficient, would be well worth while.

* Part II and also *Phil. Trans. Roy. Soc.*, A, **228**, 421–461, 1929 (referred to hereafter as Bakerian Lecture).

† C. H. Payne, *Harvard Bulletin*, 867 ; also *Stars of High Luminosity* (Harvard Monograph No. 3), 268.

‡ This is clear from the analysis contained in Milne, *M.N.*, 85, 750, 1925.

Now, the absorption coefficient has recently been studied in full detail by one of us ; the result is that the formula for the *atomic* absorption coefficient is *

$$\kappa_{at} = \frac{40}{\pi^4\sqrt{3}} \frac{e^2h^4}{cm_e(2\pi m_e)^{\frac{3}{2}}} \left[\sum_{r=1} \chi_r \left(1 + \frac{2{\cdot}43\chi_r}{kT}\right) x_r\right] \frac{P_e}{(kT)^{\frac{5}{2}}}, \qquad \text{I}$$

where

χ_r = rth ionization potential,
x_r = the fraction of the total number of atoms which have lost r-electrons.

Hence we have a formula of the type

$$\kappa_{at} = \text{constant} \left\{\sum \chi_r \left(1 + \frac{\omega\chi_r}{kT}\right) x_r\right\} \frac{P}{(kT)^{\frac{5}{2}}}. \qquad \text{I}'$$

The present paper is devoted to the completion of the analysis contained in Parts I and II to include the above type of formula for κ, which certainly has a greater claim to be " physical " than either of the two earlier assumptions.

Before concluding this introductory section it is necessary to point out that a start in this direction has already been made by W. H. McCrea in a recent important paper,† where he considers, on this more general basis, the properties of a two-constituent atmosphere. But he has overlooked the fact that the formula for κ is of the form I′, which enables a complete revision of the earlier work to be made with comparative ease ; also, the essentially new results of his paper appear here in precise analytical forms, which afford greater insight, even in the very special case he has considered.

2. *Preliminary Formulæ.*—The general principles have been fully detailed in Parts I and II (and also in Bakerian Lecture), and it is not proposed to go into detail here beyond what is necessary to make the present paper readable without too much reference to the earlier parts. But it is assumed that the reader is familiar with the methods and principles of calculation developed in Parts I and II.‡

Let ρ be the density at any level, ν the number of atoms of the given element per cm.^{3} at the level, ϵ their proportion by mass, m the atomic mass. Then

$$m\nu = \epsilon\rho.$$

Let h be the depth measured inwards from a suitable reference level. The equation of mechanical equilibrium with the neglect of radiation pressure is

$$\frac{dp}{dh} = g\rho. \qquad (1)$$

Let x_0, x_1, x_2, etc., be the fractions of neutral atoms, once-ionized atoms, etc., of the given element at any level. Let N be the total number of atoms of

* S. Chandrasekhar, " The Stellar Coefficients of Absorption and Opacity—Part II " (*Proc. Roy. Soc.*, A (in press)).

† W. H. McCrea, *M.N.*, **91**, 836, 1931, " On Model Stellar Atmospheres."

‡ A convenient reference where a very neat and concise account of the theory can be found without much detail is A. Pannekoek's article in the *Handbuch der Astrophysik*, Band iii, 1, pp. 334–340.

this element per cm.^{2} down to level h, N_0 be the number of neutral atoms down to h, N_1 that of once-ionized atoms, and so on. Then we easily obtain by an application of (1),

$$N = \frac{\epsilon p}{mg}, \tag{2}$$

$$N_0 = \frac{\epsilon}{mg}\int_0^p x_0 dp, \tag{3}$$

$$N_1 = \frac{\epsilon}{mg}\int_0^p x_1 dp, \tag{4}$$

and so on. Let P be the electron pressure at level h. The typical ionization formula may be written as

$$\frac{x_1}{x_0}P = K_1, \tag{5}$$

where K_1 is a function of temperature T alone given by

$$K_1 = \frac{q_1}{q_0}\frac{2(2\pi m_e)^{\frac{3}{2}}}{h^3}(kT)^{\frac{5}{2}}e^{-\chi_1/kT}. \tag{5'}$$

The general method is to express p, x_0, x_1, etc., in terms of P, K_1, K_2, etc., and then to evaluate the integrals (3), (4), etc., taking T, and hence K_1, K_2, etc., constant. The expressions for p, x_0, x_1, etc., are different according to circumstances. These circumstances are classified as before into five cases, each producing its own problem. The results of the evaluation of the integrals (3), (4), etc., are quoted from Part I, though references are not given as the analysis proceeds (Section I).

The optical depth τ is calculated by means of the formula

$$\tau = \int_0^h \kappa\rho dh = \frac{1}{g}\int_0^p \kappa dp. \tag{6}$$

The precise form of κ varies according to circumstances, but it is always a special case of

$$\kappa = \frac{\text{constant}}{(kT)^{\frac{5}{2}}}\left\{\sum_{r=1}\chi_r x_r\left(1+\frac{\omega\chi_r}{kT}\right)\right\}P$$
$$= (\alpha' x_1 + \beta' x_2 + \gamma' x_3 + \ldots)P. \tag{7}$$

If now $N_0^{(s)}$ be the number of excited neutral atoms of statistical weight $q_0^{(s)}$ in state $\chi_1^{(s)}$, then

$$\frac{N_0^{(s)}}{N_0} = \frac{q_0^{(s)}}{q_0}e^{-(\chi_1-\chi_1^{(s)})/kT} = A_0^{(s)}, \tag{8}$$

say. We have similar expressions for the excited ionized atoms. It will be found that in general we have

$$N_r^{(s)} = \frac{\epsilon}{mg}\phi_1(P_0, T), \tag{9}$$

$$\tau_0 = \frac{1}{g}\phi_2(P_0, T). \tag{10}$$

The method of "maxima" consists in finding the maximum of $N_r^{(s)}$ for variation of T, P_0 being a function of T varying according to relation (10), with τ_0 and g constant.

To examine the absolute magnitude effect for a given colour type (or, less accurately, spectral type), we differentiate (9) and (10), keeping T constant, treating P_0 as a function of g related by (10), τ_0 again being treated as a constant ; we then discuss the sign of $dN_r^{(s)}/dg$.

The plan of the paper is as follows :—

In Section I the formal solutions for the five problems are given on the correct theory of the absorption coefficient.

In Section II a rather detailed discussion of the absolute magnitude effects in a two-constituent atmosphere is undertaken. A preliminary study of three-constituent atmospheres is also made here.

In Section III a general discussion of the results of Sections I and II is attempted. Numerical calculations are also presented.

In the paper following this, one of us treats the closely related problem of the physical structure of the photospheric layers on the basis of the absorption coefficient of the form (7).

Also, in Section I the results alone are stated ; details of the calculations are not given. But the following set of formulæ may be noted here for reference, as they are very frequently used in the calculations.

$$\frac{dA_0^{(s)}}{A_0^{(s)}} = \frac{\chi_1 - \chi_1^{(s)}}{kT^2} dT \,; \tag{11}$$

$$\frac{dK_1}{K_1} = \frac{\chi_1 + (\frac{5}{2})kT}{kT^2} dT \,; \tag{12}$$

$$\frac{d\alpha'}{\alpha'} = -\frac{1}{T}\left[(\lambda+1) - \frac{1}{f_1}\right] dT = -\lambda \frac{dT}{T}, \tag{13}$$

say. We have used in (13), λ for $\frac{9}{2}$ and

$$f_1 = 1 + \omega\chi_1/kT. \tag{13'}$$

Section I

3. *Problem I.—Only neutral and once-ionized atoms present. Material with a mean ionization potential equal to that of the particular atoms concerned ("one-constituent atmospheres").*

We will write here

$$x_1 = x \,; \qquad x_0 = 1 - x.$$

Then

$$\frac{P}{p} = \frac{x}{1+x} \,; \qquad \frac{xP}{1-x} = K_1.$$

Solving, we find

$$x = \frac{K_1}{P + K_1} \,; \qquad p = \frac{P(P+2K_1)}{K_1} \,.$$

Using the above relations and evaluating the integrals (3) and (4) we find

$$N = \frac{\epsilon}{mg} \frac{P(P+2K_1)}{K_1}; \tag{14}$$

$$N_0 = \frac{\epsilon}{mg} \frac{P^2}{K_1}; \tag{14'}$$

$$N_1 = \frac{\epsilon}{mg} 2P. \tag{14''}$$

In this case we have for the absorption coefficient,

$$\kappa = \frac{A}{(kT)^{\frac{3}{2}}} \chi_1 \left(1 + \frac{\omega\chi_1}{kT}\right) Px = \alpha' Px \text{ (say)}, \tag{15}$$

where the constant

$$A = \frac{40}{\pi^4 \sqrt{3}} \frac{e^2h^4}{cm_e(2\pi m_e)^{\frac{3}{2}}} \cdot \frac{1}{\text{atomic mass (in gms.)}}. \tag{15'}$$

Hence for the optical thickness we find

$$\tau = \frac{1}{g}\int_0^p \kappa dp = \frac{\alpha'}{g}\int_0^P \frac{PK_1}{P+K_1} d\left(\frac{P(P+2K_1)}{K_1}\right) = \frac{\alpha' P^2}{g}. \tag{16}$$

(a) "*Extremal*" *Problem.**—We have, putting $P = P_0$,

$$N_0^{(s)} = \frac{\epsilon}{mg} \frac{P_0^2}{K_1} A_0^{(s)} \quad \text{and} \quad \tau_0 = \frac{\alpha'}{g} P_0^2.$$

Hence

$$0 = \frac{dN_0^{(s)}}{N_0^{(s)}} = \frac{dA_0^{(s)}}{A_0^{(s)}} + \frac{2dP_0}{P_0} - \frac{dK_1}{K_1}; \tag{17}$$

$$0 = \frac{2dP_0}{P_0} + \frac{d\alpha'}{\alpha'}; \tag{18}$$

whence

$$\frac{dA_0^{(s)}}{A_0^{(s)}} = \frac{dK_1}{K_1} + \frac{d\alpha'}{\alpha'}. \tag{19}$$

Using (11), (12) and (13) we find that (19) reduces to

$$\frac{\chi_1 - \chi_1^{(s)}}{kT} = \frac{\chi_1 + \frac{5}{2}kT}{kT} - \left[(\lambda+1) - \frac{1}{f_1}\right], \tag{20}$$

where we note that

$$f_1 = 1 + (\omega\chi_1)/kT; \qquad \lambda = \tfrac{9}{2}. \tag{21}$$

Simplifying (20) we obtain

$$\frac{\chi_1^{(s)}}{kT} = (\lambda - \tfrac{3}{2}) - \frac{1}{f_1}, \tag{22}$$

* We call this the "extremal" problem since actually the solution is found to correspond to a *minimum* and not a maximum.

or, using the value of λ,

$$\frac{\chi_1^{(s)}}{kT_{min}} = 3 - \frac{1}{f_1} \sim 3 \text{ approximately.} \tag{22'}$$

It is easily verified that (22′) corresponds to a minimum of $N_0^{(s)}$. We also notice the rather surprising result that in (22′), which is the solution for our " extremal " problem, P_0 has disappeared.

(b) *Absolute Magnitude Effects.*—It follows from (14′) and (16) that

$$N_0 = \frac{\epsilon}{m} \frac{\tau_0}{\alpha' K_1}. \tag{23}$$

We see that g has disappeared. Hence there is *identically* no absolute magnitude effect for the lines of the neutral atom when the conditions are such that the first-stage ionization is just setting in. This result is entirely different from the " null effect " which had previously been predicted by Milne (*Harvard Bulletin* 870). There the effect appeared as a consequence of the high temperature and the resulting completion of ionization, or, more precisely, of the fact that on the high-temperature side of the maximum P/K_1 becomes negligible. Here, however, there is identically no absolute magnitude effect. (This result is also contained in McCrea's paper already referred to, though he does not obtain equation (23)).

We will now discuss the absolute magnitude effect for the ionized atoms. From (14″)

$$\frac{dN_1}{N_1} = -\frac{dg}{g} + \frac{dP_0}{P_0}.$$

The above equation, combined with the one obtained from (16) (by differentiating it, keeping α' and τ constant), namely

$$0 = -\frac{dg}{g} + \frac{2dP_0}{P_0},$$

yields

$$\frac{dN_1}{N_1} = -\frac{1}{2}\frac{dg}{g},$$

whence

$$N_1 \propto g^{-\frac{1}{2}}. \tag{24}$$

Thus the lines of the ionized atom show a positive absolute magnitude effect, and indeed the variation of N_1 with g is *accurately* given by the very simple formula (24). This is a new result not contained in any earlier investigation. We also note in passing that just as (23) shows an identical " null effect " for the lines of the neutral atom, so (24) shows that there is identically *no* " null effect " for the lines of the ionized atom. Equation (24) is independent of the order of magnitude of P/K_1.

4. *Problem II: All atoms once ionized. Second stage setting in. Mean second-stage ionization potential equal to that of atoms in question.*

We write here

$$x_0 = 0\,; \qquad x_1 = 1 - y\,; \qquad x_2 = y.$$

Then

$$\frac{P}{p} = \frac{1+y}{2+y}\,; \qquad \frac{y}{1-y} P = K_2.$$

Solving, we get

$$y = \frac{K_2}{P+K_2}\,; \qquad p = P \cdot \frac{2P+3K_2}{P+2K_2}\,.$$

Using these to evaluate the integrals of type (3) and (4) we find that

$$N_1 = \frac{2\epsilon}{mg}\left[\frac{P(K_2+P)}{2K_2+P} - K_2 \log\left(1+\tfrac{1}{2}\frac{P}{K_2}\right)\right]; \tag{25}$$

$$N_2 = \frac{\epsilon}{mg}\left[\frac{PK_2}{2K_2+P} + 2K_2 \log\left(1+\tfrac{1}{2}\frac{P}{K_2}\right)\right]. \tag{26}$$

Our absorption coefficient is now given by

$$\kappa = \alpha' P(1-y) + \beta' P y$$

or

$$\kappa = \frac{\alpha' P^2}{P+K_2} + \frac{\beta' P K_2}{P+K_2}, \tag{27}$$

where

$$\alpha' = \frac{A}{(kT)^{\frac{5}{2}}}\chi_1\left(1+\frac{\omega\chi_1}{kT}\right); \qquad \beta' = \frac{A}{(kT)^{\frac{5}{2}}}\chi_2\left(1+\frac{\omega\chi_2}{kT}\right), \tag{28}$$

the constant A being calculated with the use of a *mean* atomic weight of the stellar material as in (15′).

We find for the optical depth τ, on carrying out the necessary integrations,

$$\tau = \frac{\alpha' P^3}{g(P+2K_2)} + \beta' \frac{m}{\epsilon} K_2 N_1. \tag{29}$$

(a) *Maximum Problem.*—After some rather tedious algebra we find that $N_1^{(s)}$ is a maximum when P_0 is a solution of the following equation, where T is understood to refer to T_{max}, the temperature at which the line attains the maximum in the precise sense defined by Milne in Part I :

$$\begin{aligned}(\chi_2 f_2 + \theta\chi_1 f_1)\frac{\chi_2 - \chi_2^{(s)}}{\chi_2 + \tfrac{5}{2}kT} + \frac{\chi_2 kT}{\chi_2 + \tfrac{5}{2}kT}\{(\lambda+1)f_2 - 1\} \\ + \frac{\theta^3}{2(\theta+2)N_1(\theta)}\,\frac{\chi_1 kT}{\chi_2 + \tfrac{5}{2}kT}\{(\lambda+1)f_1 - 1\} \\ = \left\{\chi_2 f_2 - \frac{\theta\chi_1 f_1}{N_1(\theta)}\left[\frac{\theta}{\theta+2} - \log\left(1+\tfrac{1}{2}\theta\right)\right]\right\},\end{aligned} \tag{30}$$

where f_1 and f_2 are defined by $(1+(\omega\chi_1)/kT)$ and $(1+(\omega\chi_2)/kT)$ respectively, and

$$N_1(\theta) = \left[\frac{\theta(\theta+1)}{\theta+2} - \log\left(1+\tfrac{1}{2}\theta\right)\right] \quad \text{and} \quad \theta = \frac{P_0}{K_2}. \tag{30′}$$

(b) *Absolute Magnitude Effects.*—Performing the necessary differentiations we find, working with the variable θ,

$$\frac{dN_1}{N_1} = -\frac{dg}{g}\frac{\frac{1}{2}\theta - \log(1+\frac{1}{2}\theta)}{\left(1+\frac{\beta'}{\alpha'\theta}\right)\left[\frac{\theta(\theta+1)}{\theta+2} - \log(1+\frac{1}{2}\theta)\right]}; \tag{31}$$

$$\frac{dN_2}{N_2} = -\frac{dg}{g}\frac{2\alpha' \cdot \theta\log(1+\frac{1}{2}\theta) + \beta'\left[\frac{2(\theta+1)}{\theta}\log(1+\frac{1}{2}\theta) - 1\right]}{(\alpha'\theta+\beta')\left[\frac{\theta}{\theta+2} + 2\log(1+\frac{1}{2}\theta)\right]}; \tag{32}$$

$$\frac{N_1}{N_2}d\left(\frac{N_2}{N_1}\right) = -\frac{dg}{g}\left[\frac{\beta'\left[\frac{2(\theta+1)}{\theta}\log(1+\frac{1}{2}\theta) - 1\right]}{(\alpha'\theta+\beta')\left[\frac{\theta}{\theta+2} + 2\log(1+\frac{1}{2}\theta)\right]}\right.$$
$$\left. + \frac{\alpha'\theta^2\left[\log(1+\frac{1}{2}\theta) - \frac{\frac{1}{2}\theta}{\theta+2}\right]}{(\alpha'\theta+\beta')\left[\frac{\theta(\theta+1)}{\theta+2} - \log(1+\frac{1}{2}\theta)\right]\left[\frac{\theta}{\theta+2} + 2\log(1+\frac{1}{2}\theta)\right]}\right]. \tag{33}$$

From the properties of logarithms it is easily seen that

$$\log(1+\tfrac{1}{2}\theta) > \frac{\theta/2}{\theta+1}. \tag{34}$$

Hence we have

$$\frac{dN_1}{dg} < 0; \qquad \frac{dN_2}{dg} < 0; \qquad \frac{d}{dg}\left(\frac{N_2}{N_1}\right) < 0. \tag{35}$$

(35) represents the same scheme of signs as was found for the case $\kappa \propto P$.

5. *Problem III.—The atoms in question just becoming ionized in the presence of an excess of atoms already once ionized.* We will write here

$$x_1 = x; \qquad x_0 = 1 - x. \tag{i}$$

Also we can take

$$P = \tfrac{1}{2}p \tag{ii}$$

since the free electrons form approximately one-half of the total number of particles. We find

$$N_0 = \frac{2\epsilon}{mg}\left[P - K_1\log\left(1+\frac{P}{K_1}\right)\right]; \tag{36}$$

$$N_1 = \frac{2\epsilon}{mg}K_1\log\left(1+\frac{P}{K_1}\right). \tag{36'}$$

We assume that the absorption is mainly due to the " excess " atoms. This is consistent with assumption (ii) above. Then, if $\bar{\chi}_1$ is the ionization potential of the " excess " atoms, we have

$$\kappa = \frac{A}{(kT)^{\frac{9}{2}}}\bar{\chi}_1\left(1+\frac{\omega\bar{\chi}_1}{kT}\right) = \bar{\alpha}'P \text{ (say)}. \tag{37}$$

Hence we have for the optical thickness,

$$\tau = (\bar{\alpha}'/g)P^2. \tag{37'}$$

(a) *Maximum Problem.*—We find on carrying out the differentiations,

$$\frac{\chi_1 - \chi_1^{(s)}}{\chi_1 + \frac{5}{2}kT} + \frac{\frac{1}{2}\bar{\lambda}kT}{\chi_1 + \frac{5}{2}kT}\,\frac{\theta^2/(1+\theta)}{\theta - \log(1+\theta)} = \frac{\log(1+\theta) - \theta/1+\theta}{\theta - \log(1+\theta)}, \tag{38}$$

where

$$\bar{\lambda} = \lambda + 1 - \frac{1}{\bar{f}}; \qquad \bar{f} = 1 + \frac{\omega\bar{\chi}_1}{kT}. \tag{38'}$$

(38) is the same as the solution obtained on the assumption $\kappa \propto P/(kT)^{\frac{5}{2}}$ save that, roughly speaking, we have $\lambda + 1$ instead of λ.

(b) *Absolute Magnitude Effects.*—The solutions are the same as previously given in Part II. They are

$$\frac{dN_0}{N_0} = \frac{K_1 \log\left(1 + \frac{P_0}{K_1}\right) - \frac{\frac{1}{2}P_0(P_0 + 2K_1)}{P_0 + K_1}}{P_0 - K_1 \log\left(1 + \frac{P_0}{K_1}\right)}\,\frac{dg}{g}; \tag{39}$$

$$\frac{dN_1}{N_1} = \frac{\frac{\frac{1}{2}P_0}{P_0 + K_1} - \log\left(1 + \frac{P_0}{K_1}\right)}{\log\left(1 + \frac{P_0}{K_1}\right)}\,\frac{dg}{g}; \tag{39'}$$

$$\frac{N_0}{N_1}d\left(\frac{N_1}{N_0}\right) = \frac{\frac{1}{2}P_0\left[\frac{P_0}{P_0 + K_1} - \log\left(1 + \frac{P_0}{K_1}\right)\right]}{\left[\log\left(1 + \frac{P_0}{K_1}\right)\right]\left[P_0 - K_1 \log\left(1 + \frac{P_0}{K_1}\right)\right]}\,\frac{dg}{g}. \tag{39''}$$

Since

$$\log(1+\theta) < \frac{\theta(1 + \frac{1}{2}\theta)}{1+\theta}$$

we find

$$\frac{dN_0}{dg} < 0; \qquad \frac{dN_1}{dg} < 0; \qquad \frac{d}{dg}\left(\frac{N_1}{N_0}\right) < 0. \tag{40}$$

6. *Problem IV.—The atoms in question becoming ionized with ionization potential χ_1 in the presence of an excess of atoms becoming ionized with ionization potential $\bar{\chi}_1$.* Let $\bar{K}_1$ be the ionization constant calculated with $\bar{\chi}_1$. Let $\bar{x}$ be the degree of ionization of the " excess " atoms. Then

$$\frac{x}{1-x}P = K_1; \qquad \frac{\bar{x}}{1-\bar{x}}P = \bar{K}_1; \qquad \frac{P}{p} = \frac{\bar{x}}{1+\bar{x}}. \tag{41}$$

Solving, we find that

$$p = \frac{P(P + 2\bar{K}_1)}{\bar{K}_1}.$$

We also find that

$$N = \frac{\epsilon}{mg}\frac{P(P+2\bar{K}_1)}{\bar{K}_1}; \tag{42}$$

$$N_0 = \frac{\epsilon}{mg}\frac{1}{\bar{K}_1}\left[P^2 + 2(\bar{K}_1 - K_1)\left\{P - K_1 \log\left(1 + \frac{P}{K_1}\right)\right\}\right]; \tag{42'}$$

$$N_1 = \frac{\epsilon}{mg}\frac{K_1}{\bar{K}_1}\left[2P + 2(\bar{K}_1 - K_1)\log\left(1 + \frac{P}{K_1}\right)\right]. \tag{42''}$$

We assume that the "excess" atoms alone are responsible for the absorption, which in any case is consistent with the formula for P given in (41). Hence we have

$$\kappa = \frac{A}{(kT)^{\frac{5}{2}}}\bar{\chi}_1\left(1 + \frac{\omega\bar{\chi}_1}{kT}\right)P\bar{x} = \bar{\alpha}P\bar{x} \text{ (say)}.$$

We then obtain for the optical thickness τ (*cf.* equation (16)),

$$\tau = \alpha' P^2/g. \tag{43}$$

(a) *Maximum Problem.*—Performing the necessary differentiations we find that solution of the maximum problem is given by the equation

$$N(\theta)\frac{\chi_1 - \chi_1^{(s)}}{\chi_1 + \frac{5}{2}kT} + \frac{\theta^2(\beta\theta+1)}{\theta+1}\frac{\bar{\lambda}kT}{\chi_1 + \frac{5}{2}kT} + 2(1-\beta)\left[\frac{\theta}{\theta+1} - \log(1+\theta)\right]$$
$$= \beta\theta^2\frac{\bar{\chi}_1 + \frac{5}{2}kT}{\chi_1 + \frac{5}{2}kT} + 2\beta[\theta - \log(1+\theta)]\frac{\chi_1 - \bar{\chi}_1}{\chi_1 + \frac{5}{2}kT}, \tag{44}$$

where

$$P_0 = K_1\theta\,; \qquad \beta = K_1/\bar{K}_1\,; \tag{44'}$$

$$N(\theta) = \beta\theta^2 + 2(1-\beta)(\theta - \log(1+\theta)). \tag{44''}$$

It will be noted that the solution is not very complicated in contrast with that for the simpler laws $\kappa =$ constant and $\kappa \propto P/(kT)^{\frac{5}{2}}$. The simplification lies in the circumstance that our value for κ yields a simple expression for τ.

Also, we notice that as $\beta \to 0$, (44) $\to$ (38), the solution of Problem III.

(b) *Absolute Magnitude Effects.*—We find after performing the differentiations that

$$\frac{dN_1}{N_1} = -\frac{dg}{g}\frac{\frac{1}{2}P_0 + (\bar{K}_1 - K_1)\left[\log\left(1 + \frac{P_0}{K_1}\right) - \frac{\frac{1}{2}P_0}{P_0 + K_1}\right]}{P_0 + (\bar{K}_1 - K_1)\log\left(1 + \frac{P_0}{K_1}\right)}; \tag{45}$$

$$\frac{dN_0}{N_0} = -\frac{dg}{g}K_1(\bar{K}_1 - K_1)\frac{\frac{\theta(\theta+2)}{\theta+1} - 2\log(1+\theta)}{P_0^2 + 2(\bar{K}_1 - K_1)\left[P_0 - K_1\log\left(1 + \frac{P_0}{K_1}\right)\right]}. \tag{46}$$

We will consider the sign of (45) first. The terms independent of $\bar{K}_1$ in the numerator are

$$\tfrac{1}{2}P_0+\frac{\tfrac{1}{2}P_0K_1}{P_0+K_1}-K_1\log\left(1+\frac{P_0}{K_1}\right)=\tfrac{1}{2}K_1\left[\frac{\theta(\theta+2)}{\theta+1}-2\log(1+\theta)\right]>0. \quad (46')$$

But the terms in $\bar{K}_1$ are positive. Hence

$$\frac{dN_1}{dg}<0, \quad (46'')$$

which is the same result as was obtained on the law $\kappa \propto P$. We have for the sign of (46), noting that

$$\frac{\theta(\theta+2)}{\theta+1}>2\log(1+\theta),$$

$$\left.\begin{aligned}\frac{dN_0}{dg}>0 \quad &\text{if} \quad \bar{K}_1<K_1, \quad \textit{i.e.} \text{ if} \quad \bar{\chi}_1>\chi_1\\ \frac{dN_0}{dg}<0 \quad &\text{if} \quad \bar{K}_1>K_1, \quad \textit{i.e.} \text{ if} \quad \bar{\chi}_1<\chi_1\end{aligned}\right\} \quad (47)$$

or in words, *the lines of the neutral atoms should show a positive or a negative magnitude effect according as the first ionization potential $\bar{\chi}_1$ of the excess atoms (which are also in their first-stage ionization and are present along with the atoms in question) is less than or greater than the first ionization potential of the atoms in question.*

Now McCrea (*loc. cit.*), who has considered the absolute magnitude effect by more or less numerical methods, has found that his " X-atoms " with an ionization potential of 6 volts, when surrounded by excess of hydrogen atoms with an ionization potential of 13·54 volts, show a negative absolute magnitude effect ($dN_0/dg>0$) when the hydrogen atoms are also responsible for the absorption. This is precisely what our analysis shows (equation (47)). It has to be emphasized, however, that the results of this section are *not* true if the atoms in question are merely surrounded by an excess of atoms in the first-stage ionization, but are true only if these excess atoms are also responsible for at least the major part of the general absorption. The negative magnitude effect (for his " X-atoms "), discovered by McCrea, is precisely under the conditions of this problem. It does not seem possible, however, that we have here an explanation of the negative magnitude effect which the hydrogen lines are known to show in the hotter stars.*

Again, we see from (46) that on neglecting K_1 compared with $\bar{K}_1$, which will be true for hydrogen in the cooler stars, we have approximately

$$\frac{dN_0}{N_0}=-\frac{\tfrac{1}{2}P_0\bar{K}_1}{\tfrac{1}{2}P_0^2+P_0\bar{K}_1}\frac{dg}{g}. \quad (48)$$

* This is what McCrea seems to suggest. His remarks, however, in this connection are much too vague.

Hence the order of magnitude of dN_0/N_0 is obtained by putting $P_0 = \bar{K}_1$ in the above equation ; we then find that

$$\frac{dN_0}{N_0} = -\tfrac{1}{3}\frac{dg}{g},$$

whence

$$N_0 \propto g^{-\frac{1}{3}}. \tag{49}$$

It may be noted here that the corresponding formula on the basis of $\kappa \propto P$ was

$$N_0 \propto g^{-\frac{1}{2}}. \tag{50}$$

7. *Problem V.—Three successive stages of ionization present. Material with mean first- and second-stage ionization potentials equal to those of the particular atoms concerned.*

This is the problem of the principal lines of the ionized atom. We write here

$$x_1 = x\,; \qquad x_2 = y\,; \qquad x_0 = 1 - x - y\,;$$

we have further

$$\frac{x}{1-x-y}P = K_1\,; \qquad \frac{y}{x}P = K_2.$$

Solving, it is found that

$$x = \frac{1}{1 + (P/K_1) + (K_2/P)}\,; \qquad y = \frac{K_2/P}{1 + (P/K_1) + (K_2/P)}. \tag{51}$$

We find on evaluating the integrals for N_0, N_1, and N_2 that

$$N_0 = \frac{\epsilon}{mg}\,\frac{P^3}{K_1(P+2K_2)}, \tag{52}$$

$$N_1 = \frac{2\epsilon}{mg}\left[\frac{P(P+K_2)}{P+2K_2} - K_2 \log\left(1 + \tfrac{1}{2}\frac{P}{K_2}\right)\right], \tag{52'}$$

$$N_2 = \frac{\epsilon}{mg}\left[\frac{PK_2}{P+2K_2} + 2K_2 \log\left(1 + \tfrac{1}{2}\frac{P}{K_2}\right)\right]. \tag{52''}$$

The absorption coefficient is now given by

$$\begin{aligned}\kappa &= \frac{A}{(kT)^{\frac{9}{2}}}\left[\chi_1\left(1 + \frac{\omega\chi_1}{kT}\right)x + \chi_2\left(1 + \frac{\omega\chi_2}{kT}\right)y\right]P \\ &= \alpha' Px + \beta' Py \text{ (say)} \\ &= \frac{\alpha' P}{1 + (P/K_1) + (K_2/P)} + \frac{\beta' K_2}{1 + (P/K_1) + (K_2/P)}.\end{aligned} \tag{53}$$

Hence

$$\tau = \frac{1}{g}\int_0^p \kappa dp = \alpha'\frac{m}{\epsilon}K_1 N_0 + \beta'\frac{m}{\epsilon}K_2 N_1. \tag{54}$$

It will be noted that *algebraically* the expression (54) for τ is the same as that obtained in Problem II (see equation (29)).

(a) *Maximum Problem.*—On carrying out the necessary differentiations

it is found that N_1 is a maximum when P_0 is the root of the following equation when T equals T_{max}, the temperature at which the principal lines attain the " maximum."

$$\frac{\chi_2+\frac{5}{2}kT}{kT}\left[\frac{\chi_1 f_1\theta^3}{\theta+2}+N(\theta)(\chi_2 f_2-\chi_1 f_1\theta)\right]$$
$$=\frac{\chi_1\theta^3}{2(\theta+2)}[(\lambda+1)f_1-1]+\chi_2 N(\theta)[(\lambda+1)f_2-1], \quad (55)$$

where

$$N(\theta)=\frac{\theta(\theta+1)}{\theta+2}-\log(1+\tfrac{1}{2}\theta); \qquad P_0=K_2\theta; \quad (55')$$

f_1 and f_2 having their usual meanings.

(b) *Absolute Magnitude Effects.*—The calculations to discuss the absolute magnitude effect for the atoms in the first- and second-stage ionizations are exactly the same as for Problem II. We have then

$$\frac{dN_1}{dg}<0; \qquad \frac{dN_2}{dg}<0. \quad (56)$$

Similarly, to discuss the absolute magnitude effect for the lines of the neutral atom we easily derive that

$$\left(1+\frac{\alpha'}{\beta'}\frac{P}{K_2}\right)dN_0=-\frac{dg}{g}\left(N_0-\frac{N_1P}{K_1}\right). \quad (57)$$

To discuss the sign of the right-hand side of (57) we note that

$$N_0=\frac{\epsilon}{mg}\frac{1}{K_1}\int_0^P\frac{P}{1+(P/K_1)+(K_2/P)}d\Psi, \quad (58)$$

$$N_1=\frac{\epsilon}{mg}\int_0^P\frac{1}{1+(P/K_1)+(K_2/P)}d\Psi, \quad (58')$$

where

$$\Psi=P\cdot\frac{3K_2+2P+(P^2/K_1)}{2K_2+P}.$$

Hence the sign of $N_0-(N_1P)/K_1$ is the same as that of

$$\int_0^P\frac{P}{1+(P/K_1)+(K_2/P)}d\Psi-P\int_0^P\frac{1}{1+(P/K_1)+(K_2/P)}d\Psi, \quad (59)$$

which is clearly negative. Hence by (57) we have

$$\frac{dN_0}{dg}>0. \quad (60)$$

On the assumption $\kappa\propto P$ the sign of dN_0/dg was indefinite (*cf.* Part II, equation (59)). But the *theory now quite definitely predicts a negative magnitude effect for the lines of the neutral atom, but a positive effect for the lines of the atoms in the first- and the second-stage ionizations.*

8. *Problem Va.—Three stages of ionization of the particular atoms considered in the presence of an excess of atoms already once ionized.*

We have now $$p = 2P. \tag{61}$$

We assume that $(K_1/K_2)^{\frac{1}{2}}$ is large, which is legitimate since $\chi_2 > \chi_1$, and we have the following approximations (Part I, Equations (52), (52′), (52″)).

$$N_0 = \frac{\epsilon}{mg}\frac{P_2}{K_1}; \tag{62}$$

$$N_1 = \frac{\epsilon}{mg}\left[2P - 2K_2 \log\left(1 + \frac{P}{K_2}\right) - \frac{P^2}{K_1}\right], \tag{62'}$$

$$N_2 = \frac{\epsilon}{mg} 2K_2 \log\left(1 + \frac{P}{K_2}\right). \tag{62''}$$

We assume that the " excess " atoms alone are responsible for the absorption. Then we have for the optical thickness (*cf.* Problem III, Equations (37) and (37))

$$\tau = \bar{\alpha}' P^2/g. \tag{63}$$

(a) *Maximum Problem.*—Performing the necessary differentiations, we find that for the maximum problem corresponding to $N_1^{(s)}$ the solution is given by the equation

$$\frac{\chi_2 - \chi_2^{(s)}}{kT}\left[\theta - \log(1+\theta) - \theta^2\frac{\delta}{2}\right] + \left[\frac{1}{\theta+1} - \delta\right]\frac{\bar{\lambda}}{2}\theta^2$$
$$= \left[\log(1+\theta) - \frac{\theta}{1+\theta}\right]\frac{\chi_2 + \frac{5}{2}kT}{kT} - \theta^2\frac{\delta}{2}\cdot\frac{\chi_1 + \frac{5}{2}kT}{kT}, \tag{64}$$

where

$$P = K_2\theta; \qquad \delta = K_2/K_1; \qquad \bar{\lambda} = \lambda + 1 - (1/f).$$

When we are concerned with the principal lines of the atom then the excitation potential $(\chi_2 - \chi_2^{(s)})$ is zero and the solution for the maximum problem is

$$\left[\log(1+\theta) - \frac{\theta}{1+\theta}\right]\frac{\chi_2 + \frac{5}{2}kT}{kT} - \frac{\delta}{2}\frac{\chi_1 + \frac{5}{2}kT}{kT}\theta^2 = \left(\frac{1}{\theta+1} - \delta\right)\theta^2\frac{\bar{\lambda}}{2}. \tag{64'}$$

Remembering that under the ordinary circumstances $\delta << 1$ we have corresponding to (64′) the approximate equation

$$\frac{\chi_2 + \frac{5}{2}kT}{kT} = \frac{\bar{\lambda}\theta^2/1+\theta}{2[\log(1+\theta) - \theta/1+\theta]}. \tag{65}$$

(b) *Absolute Magnitude Effects.*—On going through the usual algebra we find that

$$\frac{dN_0}{dg} \equiv 0. \tag{66}$$

$$\frac{dN_1}{dg} = -\frac{\epsilon}{mg^2}\left[\frac{P_0(P_0+2K_2)}{P_0+K_2} - 2K_2\log\left(1 + \frac{P_0}{K_2}\right)\right]. \tag{67}$$

$$\frac{dN_2}{dg} = -\frac{\epsilon}{mg^2}2K_2\left[\log\left(1 + \frac{P_0}{K_2}\right) - \frac{\frac{1}{2}P_0}{P_0+K_2}\right]. \tag{68}$$

The quantities in [ ] in (67) and (68) are both easily seen to be positive. Hence

$$\frac{dN_0}{dg} \equiv 0 ; \qquad \frac{dN_1}{dg} < 0 ; \qquad \frac{dN_2}{dg} < 0. \tag{68'}$$

This problem predicts the same " magnitude effects " for N_1 and N_2 as Problem V, but we have now for N_0 a " null effect," while there we had a negative absolute magnitude effect.

Section II

This section is devoted to the discussion of the absolute magnitude effects in a two-constituent atmosphere and a preliminary study of three-constituent atmospheres.

9. *Problem VI.—Two-constituent atmospheres. Both the types of atoms in the first-stage ionization. Discussion of the absolute magnitude effects.* This is obviously an accurate treatment of Problem IV. There the atoms in question were assumed to be surrounded by an *excess* of atoms also in the first-stage ionization, *the excess atoms also contributing to the major part of the absorption.* The analysis yielded a negative magnitude effect in one special case. But this result cannot obviously be applied to the case of the hydrogen lines in the hotter stars which are known to show a negative absolute magnitude effect (a fact to which Miss Payne has often drawn attention), at least for this, that according to current views—Rosseland, Eddington, Russell, Unsöld—hydrogen is the most abundant element in stellar atmospheres. We study therefore the problem of the two-constituent atmospheres a little more closely.

The integrated ionization formulæ for a two-constituent atmosphere have been given by Russell * and by McCrea. But we treat the whole problem *de novo.*

The notation we use is the following :—

ν = *total* number of atoms per cm.^{3} at a definite level,
$\epsilon_1\nu$ = total number of atoms per cm.^{3} of type 1 at the same level,
$\epsilon_2\nu$ = total number of atoms per cm.^{3} of type 2 at the same level,
x_1 = the degree of ionization of atoms of type 1,
x_2 = the degree of ionization of atoms of type 2,
χ_1 = the ionization potential of atoms of type 1,
χ_2 = the ionization potential of atoms of type 2,
$\bar{m}$ = the mean atomic mass,
$= \epsilon_1 m_1 + \epsilon_2 m_2$.

It will be noted that we have changed the meaning of " ϵ." In Section I ϵ denoted the proportion by *mass.* Here, however, ϵ_1 and ϵ_2 are the " abundances " of the two types of atoms with reference to the *number.* ϵ_1 and ϵ_2 are assumed to be constants down the column.

* H. N. Russell, *Astrophysical Journal*, **70**, 10, 1929, § 18 of this paper.

The hydrostatic equation is

$$\frac{dp}{dh}=g\rho, \tag{69}$$

where

$$\rho=\nu\bar{m}\,;$$

$$p=\{\epsilon_1(1+x_1)+\epsilon_2(1+x_2)\}\nu kT. \tag{70}$$

The electron pressure, P, is given by

$$P=(\epsilon_1 x_1+\epsilon_2 x_2)\nu kT\,; \tag{71}$$

x_1 and x_2 are of course given by

$$\frac{x_1}{1-x_1}P=K_1\,; \qquad \frac{x_2}{1-x_2}P=K_2, \tag{72}$$

K_1 and K_2 being the ionization constants (5′) calculated with χ_1 and χ_2 respectively. The total number of neutral atoms $N_0^{(1)}$ and $N_0^{(2)}$ of the two types are given by

$$N_0^{(1)}=\epsilon_1\int_0^h(1-x_1)\nu dh, \tag{73}$$

$$N_0^{(2)}=\epsilon_2\int_0^h(1-x_2)\nu dh. \tag{74}$$

We find after some reductions that

$$\begin{aligned}N_0^{(1)}&=\frac{\epsilon_1}{\bar{m}g}\int_0^P\frac{P}{P+K_1}d\left[\left\{1+\frac{\epsilon_1K_1}{P+K_1}+\frac{\epsilon_2K_2}{P+K_2}\right\}\frac{P(P+K_1)(P+K_2)}{P(\epsilon_1K_1+\epsilon_2K_2)+K_1K_2}\right]\\&=\frac{1}{g\bar{m}}\left[\frac{\epsilon_1P^2(P+K_2)}{P(\epsilon_1K_1+\epsilon_2K_2)+K_1K_2}-\frac{\epsilon_1PK_1}{\epsilon_1K_1+\epsilon_2K_2}+\epsilon_1P\right.\\&\quad\left.+\epsilon_2K_1\log\frac{P+K_1}{K_1}-\frac{\epsilon_2K_1K_2^2}{(\epsilon_1K_1+\epsilon_2K_2)^2}\log\frac{P(\epsilon_1K_1+\epsilon_2K_2)+K_1K_2}{K_1K_2}\right]\\&=\frac{1}{g\bar{m}}\Psi_1\text{ (say).}\end{aligned} \tag{75}$$

Similarly

$$\begin{aligned}N_0^{(2)}&=\frac{1}{g\bar{m}}\left[\frac{\epsilon_2P^2(P+K_1)}{P(\epsilon_1K_1+\epsilon_2K_2)+K_1K_2}-\frac{\epsilon_2PK_2}{\epsilon_1K_1+\epsilon_2K_2}+\epsilon_2P\right.\\&\quad\left.+\epsilon_1K_2\log\frac{P+K_2}{K_2}-\frac{\epsilon_1K_1^2K_2}{(\epsilon_1K_1+\epsilon_2K_2)^2}\log\frac{P(\epsilon_1K_1+\epsilon_2K_2)+K_1K_2}{K_1K_2}\right]\\&=\frac{1}{g\bar{m}}\Psi_2\text{ (say).}\end{aligned} \tag{76}$$

We note here that

$$d\Psi_1=\frac{\epsilon_1P}{P+K_1}d\Theta, \tag{77}$$

$$d\Psi_2=\frac{\epsilon_2P}{P+K_2}d\Theta, \tag{78}$$

where

$$\Theta = \left\{ 1 + \frac{\epsilon_1 K_1}{P+K_1} + \frac{\epsilon_2 K_2}{P+K_2} \right\} \frac{P(P+K_1)(P+K_2)}{P(\epsilon_1 K_1 + \epsilon_2 K_2) + K_1 K_2}.$$

Our mass absorption coefficient now is

$$\kappa = \frac{\epsilon_1 \nu \alpha' P x_1 + \epsilon_2 \nu \beta' P x_2}{(\epsilon_1 m_1 + \epsilon_2 m_2)\nu}$$

$$= (\epsilon_1 \alpha' P x_1 + \epsilon_2 \beta' P x_2)/\bar{m}, \tag{79}$$

where

$$\alpha' = \frac{\text{constant}}{(kT)^{\frac{5}{2}}} \chi_1 f_1 ; \qquad \beta' = \frac{\text{constant}}{(kT)^{\frac{5}{2}}} \chi_2 f_2, \tag{79'}$$

where the " constant " here is given by

$$\text{" Constant "} = \frac{40}{\pi^4 \sqrt{3}} \frac{e^2 h^4}{c m_e (2\pi m_e)^{\frac{3}{2}}}. \tag{79''}$$

Hence our expression for the optical thickness is

$$\tau = \alpha' K_1 N_0^{(1)} + \beta' K_2 N_0^{(2)}, \tag{80}$$

where $N_0^{(1)}$ and $N_0^{(2)}$ are given by (75) and (76) respectively.

We do not propose to discuss here the solution of the maximum problem. It is merely a matter of algebra, and the solutions are too complicated to be given at length. We proceed to discuss the absolute magnitude effect for the lines of the neutral atom of type 1. The results for type 2 are obtained by merely interchanging the suffixes 1 and 2.

Differentiating (75) and using (77) we get

$$dN_0^{(1)} = -\frac{1}{g^2 \bar{m}} \Psi_1 dg + \frac{1}{g\bar{m}} \frac{\epsilon_1 P_0}{P_0 + K_1} d\Theta. \tag{81}$$

Keep τ constant and differentiate (80). We have then, using (78),

$$0 = \alpha' K_1 dN_0^{(1)} + \beta' K_2 dN_0^{(2)}$$

$$= \alpha' K_1 dN_0^{(1)} + \beta' \frac{K_2}{g\bar{m}} \frac{\epsilon_2 P_0}{P_0 + K_2} d\Theta - \beta' \frac{K_2}{g^2 \bar{m}} \Psi_2 dg. \tag{81'}$$

Eliminating $d\Theta$ between (81) and (81') we get

$$dN_0^{(1)} \{ (P_0 + K_1)\epsilon_2 \beta' K_2 + (P_0 + K_2)\epsilon_1 \alpha' K_1 \}$$

$$= -\frac{dg}{g^2} \frac{\beta' K_2}{\bar{m}} \{ (P_0 + K_1)\epsilon_2 \Psi_1 - (P_0 + K_2)\epsilon_1 \Psi_2 \}. \tag{82}$$

(82) can be rewritten in the form

$$dN_0^{(1)} \left\{ 1 + \frac{P_0 + K_2}{P_0 + K_1} \frac{\epsilon_1}{\epsilon_2} \frac{\alpha'}{\beta'} \frac{K_1}{K_2} \right\} = -\frac{dg}{g} \left\{ N_0^{(1)} - N_0^{(2)} \frac{\epsilon_1}{\epsilon_2} \frac{P_0 + K_2}{P_0 + K_1} \right\}. \tag{83}$$

Hence the sign of the absolute magnitude effect depends on the sign of the terms in { } on the right-hand side of (83). Thus

$$\frac{dN_0^{(1)}}{dg} \text{ is } \left\{\begin{matrix} >0 \\ \text{or} \\ <0 \end{matrix}\right\} \text{ according as } \left\{\begin{matrix} N_0^{(1)} < N_0^{(2)}\dfrac{\epsilon_1}{\epsilon_2}\dfrac{P_0+K_2}{P_0+K_1} \\ \text{or} \\ N_0^{(1)} > N_0^{(2)}\dfrac{\epsilon_1}{\epsilon_2}\dfrac{P_0+K_2}{P_0+K_1} \end{matrix}\right\}.$$

To find the precise algebraic conditions for a positive or a negative magnitude effect we have to discuss the sign of

$$(P_0+K_1)\epsilon_2\Psi_1-(P_0+K_2)\epsilon_1\Psi_2, \tag{84}$$

which by (75) and (76) is the same as that of

$$(P_0+K_1)\int_0^{P_0}\frac{P}{P+K_1}d\Theta-(P_0+K_2)\int_0^{P_0}\frac{P}{P+K_2}d\Theta, \tag{85}$$

which is the same as

$$P_0\int_0^{P_0}P\left(\frac{1}{P+K_1}-\frac{1}{P+K_2}\right)d\Theta+\int_0^{P_0}\left(\frac{K_1P}{P+K_1}-\frac{K_2P}{P+K_2}\right)d\Theta$$
$$=(K_2-K_1)\left\{P_0\int_0^{P_0}\frac{Pd\Theta}{(P+K_1)(P+K_2)}-\int_0^{P_0}\frac{P^2d\Theta}{(P+K_1)(P+K_2)}\right\}. \tag{86}$$

Now the quantity in { } in (86) is clearly positive. Hence we have finally

$$\left.\begin{aligned} &\frac{dN_0^{(1)}}{dg}<0 \quad \text{if} \quad K_2>K_1 \quad \text{or} \quad \chi_1>\chi_2 \\ &\frac{dN_0^{(1)}}{dg}>0 \quad \text{if} \quad K_2<K_1 \quad \text{or} \quad \chi_1<\chi_2 \end{aligned}\right\}. \tag{87}$$

(83) leads to the rather surprising result that the *sign* of the absolute magnitude effect of the lines of the neutral atom is *independent* of the relative abundance of the two kinds of atoms. We state our result as follows :—

In a two-constituent stellar atmosphere, if the atoms in question are in the first-stage ionization and are present along with a second type of atom, also in the first-stage ionization, then the lines of neutral atoms of the given type should show a positive ($dN_0/dg<0$) *or a negative* ($dN_0/dg>0$) *absolute magnitude effect, according as the ionization potential of the atoms under consideration is greater than, or less than, the ionization potential of the surrounding atoms. The sign of the absolute magnitude effect is independent of abundance factors.*

Again, we cannot hail this as suggesting the necessarily correct explanation of the " anomalous " behaviour of the hydrogen lines in the hotter stars. For, though it is a fact that at high temperatures hydrogen is surrounded by atoms like helium with ionization potentials greater than 13·5 volts, yet we have to prove that in these circumstances the opposite effect caused by atoms surrounding hydrogen, but with ionization potentials less than 13·5 volts, is negligible. Hence, so long as we have not made a study of the

absolute magnitude effects in a " three-constituent " atmosphere, we cannot say anything definitely. A preliminary study of this problem is made in § 11.

10. *Problem VII.—Two-constituent atmospheres. The atoms in question (in their first stage of ionization) in the presence of neutral atoms.* We continue to use the notation of § 9. Let ϵ_1 be the fraction of the total number of atoms (both types being counted) which are of the kind under consideration. Let these be in their first stage of ionization. We then easily find that

$$p=\frac{P+(1+\epsilon_1)K_1}{\epsilon_1 K_1}P\,; \qquad x=\frac{K_1}{P+K_1}. \tag{88}$$

On solving the integrals for N_1 and N_2 we easily obtain

$$N=\frac{1}{\bar{m}g}\left[\frac{P^2}{K_1}+(1+\epsilon_1)P\right]; \tag{89}$$

$$N_1=\frac{1}{\bar{m}g}\left[2P-(1-\epsilon_1)K_1\log\left(1+\frac{P}{K_1}\right)\right]; \tag{90}$$

$$N_0=\frac{1}{\bar{m}g}\left[\frac{P^2}{K_1}-(1-\epsilon_1)P+(1-\epsilon_1)K_1\log\left(1+\frac{P}{K_1}\right)\right]. \tag{91}$$

For the absorption coefficient we clearly have

$$\kappa=\epsilon_1 a' Px/\bar{m}. \tag{92}$$

For the optical thickness we then find

$$\tau=a'K_1N_0. \tag{93}$$

We will now discuss the absolute magnitude effects. (93) shows that for the lines of the neutral atom we have an identical " null effect." To discuss the absolute magnitude effect of N_1 we easily derive, by the usual procedure,

$$dN_1=-\frac{dg}{g}\left[N_1-N_0\frac{K_1}{P_0}\right], \tag{94}$$

or, using (89′) and (89″),

$$P_0dN_1=-\frac{dg}{\bar{m}g^2}\left[P_0^2+(1-\epsilon_1)K_1(P_0+K_1)\left\{\frac{P_0}{P+K_1}-\log\left(1+\frac{P_0}{K_1}\right)\right\}\right]. \tag{95}$$

To find the sign of $[N_1-(N_0K_1)/P_0]$ we note that this is the same as that of

$$P_0\int_0^{P_0}\frac{dp}{P+K_1}-\int_0^{P_0}\frac{Pdp}{P+K_1}, \tag{96}$$

which is clearly positive. Hence

$$\frac{dN_1}{dg}<0. \tag{97}$$

We will consider two special cases.

Case I.—($\epsilon_1 \sim 1$). Then by (91) and (95) we have approximately

$$N_1 = \frac{2P_0}{\bar{m}g}, \tag{98}$$

$$P_0 dN_1 = -\frac{dg}{\bar{m}g^2} P_0^2. \tag{99}$$

We can rewrite (99) as

$$\frac{dN_1}{N_1} = -\frac{1}{2}\frac{dg}{g},$$

whence

$$N_1 \propto g^{-\frac{1}{2}}. \tag{100}$$

Thus we have identically no "null effect." (100) is in fact the result previously obtained in Problem I.

Case II.—($\epsilon_1 \sim 0$). We now have approximately

$$P_0 dN_1 = -\frac{dg}{\bar{m}g^2}(P_0 + K_1)\left[P_0 - K_1 \log\left(1 + \frac{P_0}{K_1}\right)\right], \tag{101}$$

$$N_1 = \frac{1}{\bar{m}g}\left[2P_0 - K_1 \log\left(1 + \frac{P_0}{K_1}\right)\right]. \tag{102}$$

Let us now assume that we are considering the absolute magnitude effect at such high temperatures that $(P_0/K_1) << 1$. We then have as approximate expressions,

$$N_1 \sim P_0/\bar{m}g\,; \qquad dN_1 \sim -P_0 dg/2\bar{m}g^2, \tag{103}$$

whence

$$N_1 \propto g^{-\frac{1}{2}}. \tag{104}$$

Thus in this case also we have no " null effect " on the high-temperature side. On the earlier assumption of $\kappa \propto P$, on the high-temperature side of the maximum, the theory always predicted a " null effect." But now on the basis of a purely physical theory of the absorption coefficient we have come across one special case where there is no " null effect " of this kind.

Now, Miss Payne * has drawn specific attention to the fact that " there is an almost universal residual strengthening with absolute magnitude far above the maximum for which the theory does not provide." Miss Payne's discussion is, of course, based on the earlier work.

But here we have a definite indication on purely theoretical grounds of a possibility of this phenomenon. We do not suggest that this *is* the explanation. It appears, however, that an explanation for this " residual strengthening " is not beyond the scope of the theory.

11. *Problem VIII.—Three-constituent atmospheres. All the atoms in the first-stage ionization. Discussion of the absolute magnitude effects.*

The problem of the three-constituent atmospheres appears to us as a very fundamental problem in the theory of stellar atmospheres. We do not,

* *Stars of High Luminosity*, p. 268.

however, propose to discuss this problem in detail here. We include a preliminary study here for reasons which will be apparent when we come to examine the physical significance of the results of Problem VI in § 15.

Let (i) $\epsilon_1, \epsilon_2, \epsilon_3$ be the abundances of the three types of atoms with reference to number; (ii) $\bar{m}$ be the mean atomic mass defined as equal to $\epsilon_1 m_1 + \epsilon_2 m_2 + \epsilon_3 m_3$. For the rest, the notation is the same as that of Problem VI, the suffixes 1, 2 and 3 referring to the three types of atoms respectively.

Instead of (70) we have now

$$p = \{\epsilon_1(1+x_1) + \epsilon_2(1+x_2) + \epsilon_3(1+x_3)\}\nu kT, \tag{105}$$

and similarly for the electron pressure we have

$$P = (\epsilon_1 x_1 + \epsilon_2 x_2 + \epsilon_3 x_3)\nu kT, \tag{106}$$

where, as usual,

$$x_1 = \frac{K_1}{P+K_1}; \qquad x_2 = \frac{K_2}{P+K_2}; \qquad x_3 = \frac{K_3}{P+K_3}. \tag{107}$$

We have then for the total number of the neutral atoms of the three types down to h,

$$N_0^{(1)} = \frac{\epsilon_1}{g\bar{m}}\int_0^P \frac{P}{P+K_1}d\Theta = \frac{1}{g\bar{m}}\Psi_1; \tag{108}$$

$$N_0^{(2)} = \frac{\epsilon_2}{g\bar{m}}\int_0^P \frac{P}{P+K_2}d\Theta = \frac{1}{g\bar{m}}\Psi_2; \tag{108'}$$

$$N_0^{(3)} = \frac{\epsilon_3}{g\bar{m}}\int_0^P \frac{P}{P+K_3}d\Theta = \frac{1}{g\bar{m}}\Psi_3, \tag{108''}$$

where

$$\Theta = \left\{1 + \left[\frac{\epsilon_1 K_1}{P+K_1} + \frac{\epsilon_2 K_2}{P+K_2} + \frac{\epsilon_3 K_3}{P+K_3}\right]^{-1}\right\}P. \tag{109}*$$

Our mass absorption coefficient is clearly

$$\kappa = (\epsilon_1\alpha' P x_1 + \epsilon_2\beta' P x_2 + \epsilon_3\gamma' P x_3)/\bar{m}, \tag{110}$$

where

$$\alpha' = \frac{\text{const.}}{(kT)^{\frac{5}{2}}}\chi_1 f_1; \qquad \beta' = \frac{\text{const.}}{(kT)^{\frac{5}{2}}}\chi_2 f_2; \qquad \gamma' = \frac{\text{const.}}{(kT)^{\frac{5}{2}}}\chi_3 f_3.$$

For the optical thickness we find

$$\tau = \alpha' K_1 N_0^{(1)} + \beta' K_2 N_0^{(2)} + \gamma' K_3 N_0^{(3)}. \tag{111}$$

We can now discuss the sign of the absolute magnitude effect for the lines of the neutral atoms of type 1. From (108) we have, on differentiation,

$$dN_0^{(1)} = -\frac{1}{g^2\bar{m}}\Psi_1 dg + \frac{1}{g\bar{m}}\frac{\epsilon_1 P_0}{P_0+K_1}d\Theta. \tag{112}$$

* Ψ_1, Ψ_2, Ψ_3 can easily be obtained in their analytical forms, but we do not include the formulæ here since the final expressions are too long and since also we have no occasion in the sequel to use them.

Differentiating (111), keeping τ constant, we obtain

$$0 = \alpha' K_1 dN_0^{(1)} + \beta' \frac{K_2}{g\bar{m}} \frac{\epsilon_2 P_0}{P_0 + K_2} d\Theta - \beta' \frac{K_2}{g^2 \bar{m}} \Psi_2 dg + \gamma' \frac{K_3}{g\bar{m}} \frac{\epsilon_3 P_0}{P_0 + K_3} d\Theta - \gamma' \frac{K_3}{g^2 \bar{m}} \Psi_3 dg. \tag{113}$$

Eliminating $d\Theta$ between (112) and (113), we readily obtain, after some minor reductions,

$$dN_0^{(1)} \left[1 + \frac{\alpha' K_1}{\beta' K_2 \frac{\epsilon_2}{\epsilon_1} \frac{P_0 + K_1}{P_0 + K_2} + \gamma' K_3 \frac{\epsilon_3}{\epsilon_1} \frac{P_0 + K_1}{P_0 + K_3}} \right] = -\frac{dg}{g^2 \bar{m}} \left[\Psi_1 - \frac{\beta' K_2 \Psi_2 + \gamma' K_3 \Psi_3}{\beta' K_2 \frac{\epsilon_2}{\epsilon_1} \frac{P_0 + K_1}{P_0 + K_2} + \gamma' K_3 \frac{\epsilon_3}{\epsilon_1} \frac{P_0 + K_1}{P_0 + K_3}} \right] \tag{114}$$

$$= -\frac{dg}{g} \left[N_0^{(1)} - \frac{\beta' K_2 N_0^{(2)} + \gamma' K_3 N_0^{(3)}}{\beta' K_2 \frac{\epsilon_2}{\epsilon_1} \frac{P_0 + K_1}{P_0 + K_2} + \gamma' K_3 \frac{\epsilon_3}{\epsilon_1} \frac{P_0 + K_1}{P_0 + K_3}} \right]. \tag{115}$$

As a check we note that when $\epsilon_3 = 0$ (115) reduces to (83). To discuss the sign of the quantity in [ ] in (114) we proceed exactly as in Problem VI, and find that the sign of this is the same as that of

$$\beta' \frac{\epsilon_2 K_2}{P_0 + K_2} \left[(K_2 - K_1) \left\{ P_0 \int_0^{P_0} \frac{P d\Theta}{(P + K_1)(P + K_2)} - \int_0^{P_0} \frac{P^2 d\Theta}{(P + K_1)(P + K_2)} \right\} \right] + \gamma' \frac{\epsilon_3 K_3}{P_0 + K_3} \left[(K_3 - K_1) \left\{ P_0 \int_0^{P_0} \frac{P d\Theta}{(P + K_1)(P + K_3)} - \int_0^{P_0} \frac{P^2 d\Theta}{(P + K_1)(P + K_3)} \right\} \right]. \tag{116}*$$

We have then a positive or a negative magnitude effect according as the above expression is positive or negative.

Now the quantities in the curly brackets in the two terms of (116) are both positive. Hence if

$$\left. \begin{aligned} K_1 < K_2 \quad \text{or} \quad \chi_1 > \chi_2 \\ K_1 < K_3 \quad \text{or} \quad \chi_1 > \chi_3 \end{aligned} \right\}, \tag{117}$$

we have certainly a positive effect ; or again, if

$$\left. \begin{aligned} K_1 > K_2 \quad \text{or} \quad \chi_2 > \chi_1 \\ K_1 > K_3 \quad \text{or} \quad \chi_3 > \chi_1 \end{aligned} \right\} \tag{117'}$$

we have certainly a negative effect. But the case of real physical interest is when

$$\left. \begin{aligned} K_2 < K_1 < K_3 \\ \chi_2 > \chi_1 > \chi_3 \end{aligned} \right\}, \tag{118}$$

i.e.

* (116) shows that the *sign* of the " magnitude " effect is *independent* of ϵ_1, and depends only on ϵ_2 and ϵ_3.

i.e. when the ionization potential of the atoms in question is between those of the two types of the surrounding atoms. To discuss now the sign of (116) is not so easy a matter, and we have not yet succeeded in fully analysing the conditions algebraically, but the following " qualitative " discussion yields some kind of criterion to go by.

We first note that (116), apart from constant factors, can be rewritten in the form

$$\chi_2^2\epsilon_2\left[\left(1-\frac{K_1}{K_2}\right)\left\{P_0x_2^{(0)}\int_0^{P_0}\frac{Px_2}{P+K_1}d\Theta - x_2^{(0)}\int_0^{P_0}\frac{P^2x_2}{P+K_1}d\Theta\right\}\right]$$
$$+\chi_3^2\epsilon_3\left[\left(1-\frac{K_1}{K_3}\right)\left\{P_0x_3^{(0)}\int_0^{P_0}\frac{Px_3}{P+K_1}d\Theta - x_3^{(0)}\int_0^{P_0}\frac{P^2x_3}{P+K_1}d\Theta\right\}\right]. \quad (119)$$

In writing (119) we have used the fact that β' and γ' are proportional to χ_2^2 and χ_3^2 respectively. If we now assume that χ_1 is not very widely different from χ_3 but that χ_2 differs considerably from χ_1, then $(K_1/K_2) >> 1$, but (K_1/K_3), though less than unity, is not too small. In any case the absolute values of the quantities multiplying the expressions in { } in the two terms of (119) are of the order of

$$\chi_2^2\epsilon_2K_1/K_2 \quad \text{and} \quad \chi_3^2\epsilon_3,$$

respectively; since we have also $\chi_2 > \chi_3$ we see that the " multiplying factor " for the first term is very much larger than that for the second. But since $\chi_2 > \chi_3$ it is clear that $x_2 << x_3$. If now the physical conditions are such that x_3 is practically unity down the column, we can say, roughly speaking, that the *absolute value* of the first term will be much greater than that of the second if

$$\left(\frac{\chi_2}{\chi_3}\right)^2\frac{\epsilon_2}{\epsilon_3}\frac{K_1}{K_2}(\bar{x}_2)^2 >> 1, \quad (120)$$

where $\bar{x}_2$ is some " mean " value of the ionization for atoms of the second type down the column. If we tentatively assume that $\epsilon_2 \geqslant \epsilon_3$, then all the terms on the left-hand side of (120) other than $(\bar{x}_2)^2$ are greater than unity, particularly the term (K_1/K_2), which is very large. But since we have already postulated that $\chi_2 >> \chi_3$, x_2 in general will be very small except in favourable circumstances. If, for instance, the temperature is sufficiently high to ensure x_2 being not too small in spite of $\chi_2 >> \chi_3$, then (120) will be true, and we will *then* have a *negative* absolute magnitude effect.

Finally, it may be noted that in general if we have an " n-constituent atmosphere " with the n-types of atoms occurring with the abundance factors $\epsilon_1, \epsilon_2, \ldots \epsilon_n$, and atomic masses $m_1, m_2 \ldots m_n$, and if the ionization constants are $K_1, K_2 \ldots K_n$ (calculated with the ionization potentials $\chi_1, \chi_2 \ldots \chi_n$), then we have a system of n equations of the type

$$N_0^{(l)} = \frac{\epsilon_l}{\varsigma\bar{m}}\int_0^P\frac{P}{P+K_l}d\Theta \qquad (l=1, 2 \ldots n), \quad (121)$$

where

$$\bar{m} = \Sigma\epsilon_r m_r \quad (121')$$

and

$$\Theta = \left\{1 + \left[\Sigma \frac{\epsilon_r K_r}{P + K_r}\right]^{-1}\right\} P. \qquad (121'')$$

Also, we have for the absorption coefficient and the optical thickness the following:—

$$\kappa = (\Sigma \epsilon_r a_r P x_r)/\bar{m}, \qquad (122)$$

$$\tau = \Sigma a_r K_r N_{(0)}^{(r)}, \qquad (122')$$

where

$$a_r = \frac{\text{constant}}{(kT)^{\frac{1}{2}}} \chi_r f_r. \qquad (122'')$$

In the above equations ((121'), (121''), (122) and (122')) Σ is meant to signify the summation over *all* values of the suffix r from 1 to n. The equation corresponding to (115) in the general case is

$$dN_0^{(l)}\left[1 + \frac{a_l K_l}{\sum' a_r K_r \frac{\epsilon_r}{\epsilon_l}\frac{P_0 + K_l}{P_0 + K_r}}\right] = -\frac{dg}{g}\left[N_0^{(l)} - \frac{\sum' a_r K_r N_0^{(r)}}{\sum' a_r K_r \frac{\epsilon_r}{\epsilon_l}\frac{P_0 + K_l}{P_0 + K_r}}\right], \quad (l = 1, 2, \ldots n). \qquad (123)$$

In the above equation Σ' signifies the summation over all values of the suffix r from 1 to n *except* $r = l$.

It is hardly likely that the above system of equations for the general case of an n-constituent atmosphere will ever be used in practice, but in (123) we could insert the *observed* $N_0^{(r)}$'s and the appropriate values for the K_r's and the ϵ_r's, and then *deduce numerically* the sign of the absolute magnitude effect. Of course it has to be remembered that the above system of equations is applicable only if there is no appreciable second-stage ionization in our system as a whole.

Section III

12. *Classification of Problems.*—For purposes of discussion it is convenient to have our problems grouped into different categories. (i) Problems I and VII stand by themselves and represent such simplified models that they can hardly be expected to yield any *quantitative* information regarding actual stellar atmospheres. They are more of a suggestive character and, as model one-constituent atmospheres, they have many curious properties. (ii) Problems II and V are practically identical, the latter being more general. A considerable degree of caution, however, is necessary in applying these problems to practical cases. (iii) Problems III, IV and V*a* appear to be the only class of problems which can be applied to practical examples with some degree of confidence. In the case of Problems I, II and V, since allowances are not made for the surrounding atoms in a satisfactory manner, it is hard to disentangle results which are merely mathematical consequences of the simplified nature of the model from those which belong to the intrinsic

nature of the physical problem. On the other hand, Problems III and V*a* take account of the effect of the surrounding atoms in such a general way that the physical consequences are most fully exemplified. (iv) Problems VI and VIII again fall into another class. For reasons that will be stated later, Problem VIII appears to be really *the* fundamental problem in the theory of stellar atmospheres which needs urgent attention at the present state of the theory.

We proceed now, after these preliminary remarks, to discuss in detail the separate cases.

13. *Model One-constituent Atmospheres.*—The formulæ for this problem are (*cf.* equations (14′), (14″), (16) and (23))

$$N_0 = \frac{\epsilon}{m}\frac{\tau_0}{K_1 \alpha'}; \tag{124}$$

$$N_1 = \frac{2\epsilon}{m}\sqrt{\frac{\tau_0}{\alpha' g}}; \tag{124'}$$

$$N_0^{(s)} = N_0 \frac{q_0^{(s)}}{q_0} e^{-(\chi_1 - \chi_1^{(s)})/kT}, \tag{124''}$$

where

$$\alpha' = \frac{40}{\pi^4\sqrt{3}} \frac{e^2h^4}{cm_e(2\pi m_e)^{\frac{3}{2}}} \chi_1 \left(1 + \frac{2{\cdot}43\chi_1}{kT}\right) \frac{1}{(kT)^{\frac{3}{2}} \times \text{atomic mass}}$$

$$\sim 5{\cdot}62 \times 10^{19} \frac{I_1^2}{a} \cdot \frac{1}{T^{\frac{11}{2}}}, \tag{125}$$

where

I_1 = the first ionization potential in electron volts,
a = the atomic weight (*not* mass).

Also we note that

$$K_1 = \frac{q_1}{q_0} \frac{2(2\pi m_e)^{\frac{3}{2}}}{h^3} (kT)^{\frac{5}{2}} e^{-\chi_1/kT}. \tag{126}$$

The *minimum* of $N_0^{(s)}$ occurs when

$$\chi_1^{(s)}/kT = 3, \tag{127}$$

or, expressing $\chi_1^{(s)}$ in electron volts, we have

$$T_{min} = 3{\cdot}867 \times 10^3 \times I_1^{(s)}. \tag{127'}$$

(A) *Pure Hydrogen Atmosphere.*—As a typical one-constituent atmosphere we consider first the case of a *pure* hydrogen atmosphere. We will use the above system of equations to calculate the number of hydrogen atoms capable of absorbing the Balmer lines and also the number of protons for varying physical conditions of temperature and gravity.

We have now

$\epsilon = 1$; $m = 1{\cdot}662 \times 10^{-24}$; $I_1 = 13{\cdot}54$ volts ; $I_1^{(s)} = 3{\cdot}385$ volts;
$q_1 = 1$; $q_0 = 2$; $q_0^{(s)} = 8$.

We take $\tau = 0{\cdot}613$.

Inserting the above numerical values in (124′) and (124″) we obtain

$$\log N_0^{(s)} = 2{\cdot}632 + 3 \log T + \frac{1{\cdot}705 \times 10^4}{T}, \tag{128}$$

$$\log N_1 = 12{\cdot}968 + 2{\cdot}75 \log T - \tfrac{1}{2} \log g. \tag{129}*$$

Also

$$T_{min} = 13{,}090 \text{ degrees}. \tag{130}$$

The results of the calculation are given in Table I.

TABLE I

Pure Hydrogen Atmosphere. Calculated number of Atoms per cm.² down to $\tau = 0{\cdot}613$ *capable of absorbing the Balmer Lines* ($N_0^{(s)}$), *and the number of Protons down to the same level.* (N.B.—$N_0^{(s)}$ *is independent of Surface Gravity.*)

Temperature (degrees)	3000	4000	5000	6000	7500	10,000	15,000	20,000
$\log N_0^{(s)}$ for all g	18·746	17·701	17·138	16·808	16·530	16·337	16·297	16·387
$\log N_1 + \frac{1}{2} \log g$	22·530	22·874	23·140	23·358	23·625	23·968	24·452	24·796

On comparing these values with those previously obtained in Bakerian Lecture (Table VII, also fig. 3) we note three conspicuous differences:

(i) The enormously large number of atoms predicted on the basis of this problem, particularly at low temperatures;
(ii) The occurrence of the minimum; and
(iii) The non-existence of any " null effect " for the protons.

Now Miss Payne, discussing on the basis of the results of Parts I and II, has written : " The predicted numbers of hydrogen atoms for the *cooler* stars are in glaring disagreement with observation. Giant M stars have thousands more observed than predicted ; super-giants tens of thousands." † We have not compared the above theoretical values with the observational data in detail, and it would be useless to expect that there would be quantitative agreement either. But it is of interest to note that if the stellar atmospheres of the giant M stars are mostly hydrogen, then there is a definite indication, on purely theoretical grounds, of a possibility of the occurrence of a very large number of hydrogen atoms—in fact, thousands more than that predicted on the earlier theory. Physically also it is clear why we should get this effect (in one-constituent atmospheres) at low temperatures. At low temperatures the hydrogen atoms are mostly neutral, and our theory of the absorption coefficient shows that κ should now be very small, with the result that the atmosphere becomes very transparent, and for a *given* optical thickness τ the " mass of the material above the photosphere " is necessarily large.

Again, it was pointed out in Part II that for hydrogen near the maximum, Problem I is applicable, and on this basis a formula of the type

$$N_0 \propto g^{-0{\cdot}185}$$

* In this section we will use the convention of denoting logarithms to the base 10 by " log " and the natural logarithms by " ln."

† *Stars of High Luminosity*, p. 274.

was suggested. Miss Payne writes *: " The prediction is not satisfactory ; the sign is contrary to observation." But now we note that if the conditions of Problem I apply we should have a " null effect." There is clearly now an improvement in the prediction of the theory.

(B) *Application to a "Helium Atmosphere."*—As a second example we consider a " helium atmosphere " for which Problem I is applicable. We assume a 5 per cent. abundance, and calculate with the following data:—

$$\epsilon = 0{\cdot}05\ ; \quad m = 4 \times 1{\cdot}662 \times 10^{-24}\ ; \quad I_1 = 24{\cdot}47 \text{ volts}\ ; \quad I_1^{(s)} = 3{\cdot}48 \text{ volts};$$

we again take

$$\tau = 0{\cdot}613.$$

The statistical weights used are those of orthohelium.†

The formulæ corresponding to (128) and (129) are

$$\log N_0^{(s)} = 0{\cdot}562 + 3 \log T + \frac{1{\cdot}753 \times 10^4}{T}; \tag{131}$$

$$\log N_1 = 11{\cdot}109 + 2{\cdot}75 \log T - \tfrac{1}{2} \log g. \tag{132}$$

Also, the minimum of $N_0^{(s)}$ occurs at

$$T_{min} = 13{,}460 \text{ degrees}. \tag{131'}$$

The calculations are summarised in Table II.

TABLE II

Helium. Application of Problem I to a Stellar Atmosphere with a 5 per cent. (by mass) abundance of Helium. Calculated number of Atoms per cm.² down to $\tau = 0{\cdot}613$ capable of absorbing $\lambda\lambda$ 4472 ($N_0^{(s)}$), and the number of Ionized Helium Atoms (N_1).

Temperature (degrees)	5000	7500	10,000	15,000	20,000	25,000	30,000
$\log N_0^{(s)}$ for all g	15·165	14·524	14·315	14·258	14·340	14·456	14·577
$\log N_1 + \frac{1}{2} \log g$	21·281	21·766	22·109	22·593	22·937	23·203	23·421

Now Elvey,‡ who has made a detailed analysis of the contours of helium lines in stellar spectra, has previously considered precisely this problem and worked out this same example on the earlier theory, and finds that the observed number of helium atoms is thousands of times larger than that predicted by the theory. When we compare the above values (Table II) with his curves (*loc. cit.*, fig. 8) we find that there is still a discrepancy—but now by only a factor 100.§ Also, the conditions of this problem require a minimum at 13,460 degrees, and the " observed points " plotted in his fig. 8 seem to have a minimum somewhere here.

* *Stars of High Luminosity*, p. 271.

† The spectroscopic data are taken from R. H. Fowler, *Statistical Mechanics*, p. 375.

‡ C. T. Elvey, *Astrophysical Journal*, **70**, 141, 1929.

§ We have used for I in our formula for κ the observed ionization potential of 24·5 volts. But if we had adopted the usual convention of using a " net charge " of unity for ionized atoms, then we should have used $I = 13{\cdot}5$ volts and our values for $N_0^{(s)}$ would then have been four times larger.

Finally, it has to be emphasised that we do not by any means suggest that we have found a decisive and clear-cut explanation of such large numbers of the hydrogen and helium atoms as are observed. As was pointed out earlier, this phenomenon is physically intelligible if the absorption coefficient falls rapidly with decreasing ionization so as to obtain increased transparency at the lower temperatures. We can take it therefore that there is observational evidence for the absorption coefficient depending on the degree of ionization, and since our physical theory of the absorption coefficient also indicates this, we are probably right in saying that there is observational support for this aspect of the theory of the absorption coefficient.

However that may be, it is clear that in *model one-constituent atmospheres* we have, in general, a very large number of atoms predicted for a given τ, particularly at low temperatures, and we are content in merely drawing attention in this connection to a similar circumstance observed in *actual* stellar atmospheres. In any case, the results of this Problem I can hardly be regarded as anything more than of a suggestive character, but it is clear that *model* one-constituent stellar atmospheres on the present theory of the absorption coefficient have many interesting and unexpected properties.

Before proceeding to discuss the numerical solutions of the maximum problems, we dispose first of a few somewhat qualitative remarks connected with Problems III, V, VI and VIII.

14. *Absolute Magnitude Effects when Three Successive Stages of Ionization are present.*—The scheme of signs for the absolute magnitude effects obtained in connection with the various problems can be summarised as below:—

Problem	I	II	III	V	V*a*
dN_0/dg	$\equiv 0$		< 0	> 0	$\equiv 0$
dN_1/dg	< 0	< 0	< 0	< 0	< 0
dN_2/dg		< 0		< 0	< 0

Thus, that dN_1/dg is negative comes out under the conditions of all the " problems " both on this theory of the absorption coefficient and also on the earlier assumption of $\kappa \propto P$. The strengthening of the principal lines of the ionized atoms as we pass from dwarfs to giants of the same " colour " type can therefore be regarded as a fact definitely predicted by theory.

The case of interest, however, is the sign of dN_0/dg. Here the theory is completely ambiguous. We have all the possibilities—" positive," " null " and " negative " effects—occurring. But from the *observed* absolute magnitude effects we can go back and find which " problem " represents the stellar atmospheric conditions in the particular cases in hand. The fact that many lines of the neutral atoms show positive magnitude effects seems to indicate that the " atmospheric " conditions are those contemplated in Problem III. This is, on the whole, what would be expected.

15. *The " Abnormal " Behaviour of the Hydrogen Lines.*—Miss Payne has repeatedly drawn attention to an " abnormal " behaviour of the hydrogen lines.* She points out that these show a *negative* magnitude effect in the

* *Stars of High Luminosity*, p. 268.

" hotter " stars and a positive effect in the " cooler " stars. The negative effect for the hydrogen lines is found in the Ao stars at a temperature of about 10,000 degrees. We have now to inquire whether we can offer any explanation of this behaviour of the hydrogen lines.

In Problem IV we obtained a negative magnitude effect for the lines of the neutral atom if the atoms in question were surrounded by an excess of atoms responsible for the major part of the general absorption and with an ionization potential greater than that of the atoms under consideration. We cannot, however, offer this as an explanation of the " hydrogen anomaly," since it is inconceivable that hydrogen could be surrounded by an excess of atoms of another type. In Problem VI we again obtained a negative effect, and this time the effect appeared independent of abundance factors. But this again has to be rejected as not suggesting the necessarily true explanation for the behaviour of the hydrogen lines, since the effect of the surrounding atoms with ionization potentials less than that of hydrogen was not considered. With this in view, a preliminary study of the three-constituent atmospheres was undertaken in the form of Problem VIII, and the effect of the two types of atoms, one type with an ionization potential greater than, and the other type with an ionization potential less than that of the atoms in question, was considered. The resulting inequality (120) gives the condition for a negative magnitude effect.* The discussion there shows that, in favourable circumstances, a negative effect can occur for the hydrogen lines in the hotter stars, if we now make the suffixes 1, 2 and 3 correspond to hydrogen, helium (say) and " metallic vapours," respectively.

There is, however, still another place in the theory where this behaviour of the hydrogen lines could find a place. When conditions of Problem V prevail we have proved that there is a negative absolute magnitude effect for the lines of the neutral atom. It is clear that the analysis of Problem V also applies when three distinct (" mean ") ionization potentials are operative at the particular temperature concerned and the lowest of the three ionization potentials corresponds to the first ionization potential of the atoms in question. *A priori* considerations show that if the lowest of the three " mean " ionization potentials should correspond to the ionization potential of hydrogen (*i.e.* 13·5 volts), then the temperature must be really high. Since the hydrogen lines actually do show a negative magnitude effect, we can perhaps take this to indicate that in these circumstances the stellar atmospheric conditions are those which are contemplated in Problem V.

It would be very difficult to say anything more precise, in addition to what has already been said above, but it is clear that the behaviour of the hydrogen lines need no longer be called " abnormal."

16. *Numerical Applications of the Solutions of the Maximum Problems (Problem III).*—First, it has to be said that the use of the solutions of the maximum problems to test any theory of the absorption coefficient is not a wholly satisfactory procedure. A better method would be to *predict* the maxima from plausible values of κ_0 rather than deduce κ_0 from the observed

* (120) is, in fact, the criterion to judge as to when we can apply the results for a two-constituent atmosphere to the case of a three-constituent atmosphere.

maximum to test the theory. But better still would be to follow the kind of procedure adopted by Elvey in his work on the contours of helium lines. But all these processes mean very elaborate numerical work, and this is not the place for them. Meantime we proceed to discuss the solutions for the maximum problems. We begin with Problem III. The general method is as follows :

The solution for the maximum problem is (equation (38))

$$\frac{\chi_1-\chi_1^{(s)}}{\chi_1+\frac{5}{2}kT}+\frac{\frac{1}{2}\bar{\lambda}kT}{\chi_1+\frac{5}{2}kT}\frac{\theta^2/1+\theta}{\theta-ln(1+\theta)}=\frac{ln(1+\theta)-\theta/1+\theta}{\theta-ln(1+\theta)} \tag{133}$$

where T is meant to refer to T_{max}. Since χ_1, $\chi_1^{(s)}$ and T_{max} are all known, the above equation can be solved for θ. The solutions in particular cases can be found quite easily by the use of any standard four-figure mathematical tables. The subsequent calculations show that the root of (133) can be anything, and it is not advisable to obtain or use approximate solutions. In many cases the use of " approximate " solutions is definitely very misleading.

Once we determine θ from (133), the value of the pressure P_0 at the base is fixed. For $P_0=\theta K_1$, and K_1, being a function only of the temperature and χ_1, is known directly from the initial data of the problem. Now we have for the optical thickness τ,

$$\tau=\bar{\alpha}'P_0^2/g, \tag{134}$$

where, by (125),

$$\bar{\alpha}'=5{\cdot}62\times10^{19}\times(\bar{I})^2(a)^{-1}T^{-\frac{11}{2}}, \tag{135}$$

where $\bar{I}$ is the " mean " ionization potential of *all* the atoms in the first-stage ionization. The value of τ is known from the definition of the " maximum," and since g is also one of the initial data of the problem, (134) determines P_0 in terms of $\bar{I}$ and a. We have in fact,

$$P_0=\sqrt{\frac{g\tau\times T^{\frac{11}{2}}}{5{\cdot}62\times10^{19}}}\times\frac{a^{\frac{1}{2}}}{\bar{I}}=\sqrt{\frac{g\tau\times T^{\frac{11}{2}}}{5{\cdot}62\times10^{19}}}(\eta)^{-1}, \tag{136}$$

where

$$\eta=\bar{I}/a^{\frac{1}{2}}. \tag{136'}$$

Since $P_0=\theta K_1$, (136) determines η. It is the determination of this quantity η for particular special cases, and the seeing whether they are all mutually consistent among themselves and of the expected order, which will constitute the test of our theory. Of course, other slightly different methods could be adopted, but in effect they will all be equivalent to this.

A. *Application to the Maximum of the Zinc Triplet* 1^3P-1^3S.—For reasons stated in Bakerian Lecture we again choose the maximum of the zinc triplet $\lambda\lambda$ 4810, 4722, 4680 as the fundamental one. The results of the calculation are given in Table III.

The agreement between the two values of η is almost perfect. *Perfect* agreement would, however, be to a large extent accidental, but we note that they are of the expected order.

TABLE III

Maximum of Zinc Lines $1^3P - 1^3S$ $\lambda\lambda$ 4810, 4722, 4680, Type Go ($\tau = \frac{1}{3}$);
$\chi_1 = 9{\cdot}34$ volts; $\chi_1^{(s)} = 5{\cdot}33$ volts; $q_0 = 1$; $q_1 = 2$

Star	Sun	Capella
Temperature $= T_{max}$ (degrees)	5740	5200
Surface Gravity $= g$ (cm. sec.$^{-2}$)	$2{\cdot}74 \times 10^4$	$6{\cdot}08 \times 10^2$
Equilibrium Constant $= K_1$	21·6	2·32
Equation for θ	$(1+\theta)ln(1+\theta) = 0{\cdot}3680\theta(\theta + 2{\cdot}717)$	$(1+\theta)ln(1+\theta) = 0{\cdot}3623\theta(\theta + 2{\cdot}760)$
Root of Equation $= \theta$	1·26	1·325
Pressure $P_0 = \theta K_1$ (dynes cm.$^{-2}$)	27·16	3·07
P_0 as given by (136) (dynes cm.$^{-2}$)	$277\eta^{-1}$	$31{\cdot}45\eta^{-1}$
$\eta = \bar{I}/\sqrt{a}$ (volts)	10·2	10·23

At this stage it is necessary to explain the origin of the appearance of $\bar{I}$ in our theory. It comes about in this way. The absorption coefficient (κ) can be evaluated accurately only for the case of hydrogen. Strictly speaking, the formula I (§ 1) is true only when there is a single term in the summation (namely, the first), and χ corresponds to 13·54 electron-volts, the ionization potential of hydrogen. The generalisation of a result true only for hydrogen to more complicated atoms is, frankly, nothing beyond pure extrapolation. Instead of "net-charges," we use directly those derived from the known ionization potentials. This is how $\bar{I}$ appears in the general formula. Though this extrapolation is perhaps the most "rational" that could be made at present, the usual convention is to use directly for the "net-charges" integral multiples of the charge of the proton. The latter method is perhaps rather too favourable for a large absorption. The former, on the other hand, seems to cut down the value of the absorption coefficient rather drastically. Thus, for completely once-ionized calcium our formula, using the known I_1, gives only 25 per cent. of the value which would be obtained by taking directly the net charge as $+e$. It is likely that the actual value is somewhere between. In any case it is now clear that since the circumstances of Problem III correspond to nearly all the atoms being once ionized, the choice of 13·5 volts for $\bar{I}$ is not inconsistent with our theory of the absorption coefficient. At higher temperatures, when there is a considerable amount of second-stage ionization and also elements like helium getting ionized, there is good reason to expect that $\bar{I}$ is even larger than 13·5 volts; in favourable circumstances it could even be somewhere about 20 volts.

Now, to come back to the deduced values of η, we see at once that they are of the expected order, though perhaps a smaller value (5 or 6) would have been more satisfying. But as K_1 is extremely sensitive to the temperature while the value of the root of the equation for θ is not, a small increase in temperature would appreciably increase the value of P_0 calculated from the "maximum," and this would result in smaller values of η, and better "agreement" would result.

(B) *Application to the Maxima of the Balmer Lines.*—As a second example we apply the equations of Problem III to the case of the maxima of the

Balmer lines. The data are taken from Miss Payne's article (*Harvard Bulletin*, 867). The results of the calculation are given in Table IV.

TABLE IV

Maximum of Balmer Lines, $r = 0{\cdot}69$, Type A0 ;
$\chi_1 = 13{\cdot}54$ volts ; $\chi_1^{(s)} = 3{\cdot}385$ volts ; $q_1 = 1$, $q_0 = 2$

Temperature $= T_{max}$	10,000 degrees
Surface Gravity $= g$ (cm. sec.$^{-2}$)	$1{\cdot}622 \times 10^4$
Equilibrium Constant $= K_1$	$5{\cdot}13 \times 10^2$
Equation for θ	$(1+\theta)ln(1+\theta) = 0{\cdot}484\theta(\theta + 2{\cdot}06)$
Root of Equation $= \theta$	0·2
P_0 = pressure at base $= \theta K_1$ (dynes cm.$^{-2}$)	102·6
P_0 derived from (136) (dynes cm.$^{-2}$)	$1240\eta^{-1}$
$\eta = I/\sqrt{a}$	12·08

The agreement between this new value of η and those previously determined is quite good.

Though it would be rash to draw any conclusions from the above values of η, yet it is tempting to surmise that a is not very large (in which case we would obtain a rather good quantitative " agreement ") ; this implies a low mean value for the atomic weight and a consequent abundance of low atomic-weight elements, for which there appears to be evidence even otherwise.

17. *Applications of Problem Va.*—The solutions for the maximum problems are given in Section I. The procedure adopted is the same as in § 16. We apply our formulæ to Mg^+ and Ti^+. The observational data are again taken from Miss Payne's article. The relevant data and the results of calculation are given in Table V.

TABLE V

Maxima of the Lines of Ti^+ *and* Mg^+ ($r = 0{\cdot}69$)

Atom	Ti^+	Mg^+
Ionization Potential (volts)	13·6	14·97
Excitation Potential (volts)	1·16	8·83
Spectral Type	F5	A3
Temperature $= T_{max}$ (degrees)	7000	9000
Logarithm of Surface Gravity	3·30	4·36
Equilibrium Constant $= K_2$	1·332	42·74
Equation for θ	$\frac{ln(1+\theta)}{\theta+1} = 0{\cdot}1736\frac{\theta(\theta+5{\cdot}76)}{\theta+1}$	$\frac{ln(1+\theta)}{\theta+1} = 0{\cdot}4249\frac{\theta(\theta+2{\cdot}354)}{\theta+1}$
θ	9·00	0·5715
Pressure at base $P_0 = \theta K_2$ (dynes cm.$^{-2}$)	11·99	24·43
P_0 as derived from (136) (dynes cm.$^{-2}$)	$165\eta^{-1}$	$1115\eta^{-1}$
$\eta = \bar{I}/\sqrt{a}$	13·74	45·6

In the case of Ti^+ the agreement between the value of η obtained here and the values deduced earlier is as good as could be expected. The case of Mg^+ is different. The value is definitely too large.

18. *The Maxima of the Principal Lines of Ionized Calcium.*—Miss Payne

records that the principal lines of ionized calcium attain their maximum (for $r = 0{\cdot}69$) in stars of type K_2 and at a temperature of 3800 degrees. The surface gravity for the giants of type K_2 is, according to Russell, Dugan and Stewart, 4×10^4 ($\log g = 4{\cdot}60$).

We continue to apply Problem V*a*. The equation to be solved for θ is found to be

$$ln(1+\theta) = 0{\cdot}07125\frac{\theta(\theta+14{\cdot}04)}{1+\theta}. \qquad (137)$$

On solving (137) we find the surprising result that

$$\theta = 39{\cdot}07. \qquad (138)$$

It is known that in the earlier Fowler-Milne treatment of the maxima of absorption lines the value found for P from the maximum of Ca^+ lines was very small. Since now by (138) θ is very large, we find that this anomaly is to a certain extent lessened. It is not, however, removed by any means. The value of the equilibrium constant K_2 is much too small, with the result that the value of P_0 is still too small. If we compare this value of P_0 with that derived from the optical thickness with reasonable values of κ_0, we find that the discrepancy is far too large. It seems as though in this case something very fundamental has been overlooked. As was suggested earlier, since Ca^+ lines are of chromospheric origin, they must be exposed to strong selective radiation, and we ought probably to take much smaller values of g, which would tend to remove the discrepancy.

19. *Imperfections of the Theory. Concluding Remarks.*—No very exhaustive test of the theory and correlation with observational material has been undertaken. Such calculations as have been made show that our formula for the absorption coefficient gives for κ_0, at the base, values not different from those derived from the solutions of the maximum problems by more than a factor four or five. There would, however, be a better agreement if we assumed a rather low mean atomic weight, for which there seems to be evidence otherwise. Since, however, as has been pointed out earlier, the method of maxima is not a wholly satisfactory procedure to test *any* theory for the absorption coefficient, the consistency of the values of η deduced under widely different circumstances —first, among themselves, and, secondly, with the expected order—can be taken to indicate that our new formula for the absorption coefficient fits in with observed facts very satisfactorily, particularly when we note that all previous attempts to correlate theory with observation led to discrepancies of the order of hundreds or even thousands.

However, the theory as a whole suffers from serious deficiencies, and the present paper hardly improves the general setting of the theory. The imperfections discussed in Bakerian Lecture in connection with the earlier work apply equally well now. Also, of the neglected factors, selective radiation pressure needs attention.

Even apart from these, we feel doubtful if a really exhaustive test of theory on the basis of the formulæ presented here is worth undertaking ; for we do not think that the full possibilities of the theory, even as it stands,

have been utilised. This becomes clear when we compare the results of Problems I, VI and VIII (representing one-, two- and three-constituent atmospheres respectively) on the one hand, and those of Problems III and V*a* on the other. Problem I in its results is extremely suggestive and brings into the scope of the theory the non-existence of " null effects " and the possibility of a huge number of atoms corresponding to a given τ (as is observed in the case of hydrogen and helium). Problem VI again, while of a suggestive character, is hardly able to give any information without the analysis of Problem VIII. These two problems jointly offer a very plausible explanation of the negative absolute magnitude effect which the hydrogen lines are known to show in the hotter stars. These results on the one hand, and the fair amount of success with which Problems III and V*a* have been applied to practical cases on the other, make it increasingly clear that the urgent need for the theory at the present moment is an exhaustive analysis of three-constituent atmospheres. It is clear that this model is best suited to represent actual stellar atmospheric conditions. For, now we have the eight independent variables ϵ_1, ϵ_2, ϵ_3, χ_1, χ_2, χ_3, T and g, and the three auxiliary parameters K_1, K_2, K_3. By a suitable choice of ϵ_1, ϵ_2, ϵ_3 and K_1, K_2, K_3, most stellar atmospheric conditions could probably be represented. In fact, all our " problems " are only special cases of this, but in simplifying this fundamental problem we lose actual contact with facts except at a few points, with the result that we do not get a full and complete picture. Explanations of facts which are presumably closely related to each other in reality suggest themselves in the theory at distant and detached places. We are thus still very far indeed from arriving at a connected and convincing account of the facts about stellar atmospheres.

We believe therefore that a careful analysis of the properties of three-constituent atmospheres on the basis of the physical theory of the absorption coefficient will lead to a much clearer understanding of the problems associated with stellar atmospheres than has hitherto been made possible. An analysis of this fundamental problem has never been attempted before, and hardly a beginning has been made here. We have already studied this problem in some detail, but since this paper has already run to some length, and since, further, this communication as it stands marks a definite stage in the development of the theory, we reserve all further work for a future occasion.

Postscript added on January 5.—After this paper was written, a paper by Y. Sugiura * was received which modifies slightly our precise formula for the absorption coefficient I (§ 1), and this in turn affects to a small extent the numerical work contained in §§ 13, 16 and 17.

Instead of I a more " exact " formula appears to be †

$$\kappa_{at} = \frac{40}{\pi^4\sqrt{3}} \frac{e^2h^4}{cm_e(2\pi m_e)^{\frac{3}{2}}} \left[\sum_{r=1} \chi_r x_r \left(6 + \frac{2{\cdot}43\chi_r}{kT}\right)\right] \frac{P}{(kT)^{\frac{5}{2}}}, \qquad \text{II}$$

* Y. Sugiura, *Sc. Pap. I.P.C.R.* (Tokyo), **17**, 89, 1931, Communication No. 339.

† See the " Note added in proof " in Chandrasekhar's paper on " The Stellar Coefficients of Absorption and Opacity, Part II," *Proc. Roy. Soc.* (in press).

with "6" in place of "1" in equation I. The general theory (Sections I and II, and §§ 12, 14, 15, 18 and 19 of Section III) is, of course, unaffected since II is also of the form I′ (§ 1). (II is found to give values for κ only about 5 *per cent.* larger than those given by I.)

Now the formula used for κ in §§ 13, 16 and 17 is (*cf.* equation (125))

$$\kappa = \bar{\alpha}' P x,$$

with

$$\bar{\alpha}' = 5{\cdot}62 \times 10^{19} \frac{\bar{I}^2}{a} \cdot \frac{1}{T^{\frac{1}{2}}}. \qquad (125)$$

(125) as it stands is an approximate formula. It could be made exact by multiplying the right-hand side of (125) by a factor ξ, which according to our *earlier* formula (I) for κ is given by

$$\xi = 1 + \frac{kT}{2{\cdot}43\bar{\chi}} \sim 1 + 2{\cdot}53 \times 10^{-6} T. \qquad (125')^*$$

Hence we were *then* justified in taking $\xi \sim 1$. But by II we now have

$$\xi = 1 + \frac{6kT}{2{\cdot}43\bar{\chi}} \sim 1 + 1{\cdot}517 \times 10^{-5} T. \qquad (125'')$$

It is now not quite so fair to assume $\xi = 1$. Thus we have now instead of (136),

$$P_0 = \sqrt{\frac{g\tau T^{\frac{11}{2}}}{5{\cdot}62 \times 10^{19}}} \eta^{-1} \xi^{-\frac{1}{2}}, \qquad (136'')$$

with the same definition of "η" as before.

Further, our equation for θ is also changed. For in (133),

$$\bar{\lambda} = (\lambda + 1) - \bar{f}^{-1}$$

where

$$\bar{f} = 1 + \frac{\omega \bar{\chi}_1}{kT}.$$

According to I, $\omega = 2{\cdot}43$, but by II, $\omega = 0{\cdot}405$.

On the basis of these results the five numerical examples given in §§ 16 and 17 were revised, with the following results.

Revised Calculations for "η"

Atom	*Zn* (Capella)	*Zn* (Sun)	*H*	Ti^+	Mg^+
Equation for θ. $ln(1+\theta) =$	$0{\cdot}3612\frac{\theta(\theta+2{\cdot}769)}{1+\theta}$	$0{\cdot}3667\frac{\theta(\theta+2{\cdot}727)}{1+\theta}$	$0{\cdot}4841\frac{\theta(\theta+2{\cdot}066)}{1+\theta}$	$0{\cdot}1719\frac{\theta(\theta+5{\cdot}818)}{1+\theta}$	$0{\cdot}4236\frac{\theta(\theta+2{\cdot}358)}{1+\theta}$
Root, $\theta =$	1·350	1·295	0·200	9·20	0·63
Pressure at base, $P_0 = \theta K_1$ (dynes cm.$^{-2}$)	3·132	28·04	102·6	12·26	26·93
P_0 as given by (136″) (dynes cm.$^{-2}$)	30·24 η^{-1}	265·1 η^{-1}	1155 η^{-1}	156·9 η^{-1}	1046 η^{-1}
η (volts)	9·65	9·45	11·26	12·8	38·9

There is a distinct improvement in these newly derived values of η over those previously deduced. The "general agreement," however, is the same as before.

* We take $\bar{\chi}$ corresponding to 13·5 volts.

The numerical work of § 13 on the " model one-constituent atmospheres " is hardly affected. To get the more " exact " values for log $N_0^{(s)}$ and log N_1 we should subtract log ξ and $\frac{1}{2}$ log ξ from the values tabulated in § 13. These corrections are hardly appreciable. For instance, in the case of the hydrogen atmosphere the corrections for log $N_0^{(s)}$ and log N_1 at 10,000 degrees are only $-0{\cdot}06$ and $-0{\cdot}03$, respectively. At the lower temperatures the " corrections " are considerably less.

MODEL STELLAR PHOTOSPHERES.

S. Chandrasekhar.

(Communicated by Professor Milne)

1. The physical structure of the photospheric layers has been studied by Woltjer, Jeans and Milne among others.* For our purposes here the treatment given by Milne in his Bakerian Lecture is the most convenient. In the analysis Milne assumes a law $\kappa \propto P/T^{\frac{9}{2}}$, and in his numerical work to evaluate the temperature, temperature gradient, pressure, density, etc., he uses for the constant of proportionality in the law $\kappa \propto P/T^{\frac{9}{2}}$, that derived by him " astrophysically " from the maxima of zinc lines.

But McCrea † has recently pointed out that another profitable approach to the problems associated with the stellar atmospheres is to work with simplified " models " and to work out fully the consequences of the *physical theory*. It is proposed therefore to apply here this general scheme of advance to the problem of the physical structure of the photosphere on the basis of the author's recent work on the stellar absorption coefficient.‡

2. *Statement of the Problem.*—The problem is to determine the distribution of pressure, density, temperature, temperature gradient and opacity through the photospheric layers (in terms of the optical thickness τ and depth h) of a one-constituent atmosphere where the first stage ionization has set in. In the solution of this problem we make (following Milne) the simplifying assumption of using a mean value for the ionization, instead of treating this as a variable with the " depth." The formula for the absorption coefficient is (a special case of equation (63) of my paper)

$$\kappa = \frac{40}{\pi^4\sqrt{3}} \frac{e^2h^4}{cm(2\pi m)^{\frac{3}{2}}}\chi_1\left(1+\frac{2{\cdot}43\chi_1}{kT}\right)\frac{P\bar{x}}{(kT)^{\frac{9}{2}}\text{ atomic mass}}, \qquad (1)$$

* Woltjer, *B.A.N.*, **2**, 171 (No. 66), 1924 ; Jeans, *M.N.*, **85**, 199, 1925 ; Milne, *M.N.*, **85**, 768, 1925 ; Milne, *M.N.*, **90**, 17, 1929, §§ 20, 21 and 22 ; Milne, *Phil. Trans.*, A, **228**, 421, 1929, § 16 (Bakerian Lecture).

† W. H. McCrea, " Model Stellar Atmospheres," *M.N.*, **91**, 836, 1931. This kind of approach to stellar problems in *general* has been very often urged by Professor Milne on various occasions. See, for instance, his remarks in *Zeits. für Astrophysik*, **1**, 98, 1930, where he compares stellar problems to those in geometry.

‡ S. Chandrasekhar, *Proc. Roy. Soc.*, A (in press).

where P = electron pressure,
χ_1 = the first " ionization potential " in ergs,
$\bar{x}$ = the fraction of the atoms ionized.

Since for the temperatures usually met with in the photospheric layers ($\sim 10^4$ degrees) the second term in the brackets in equation (1) is by far the most important, we take as our basic formula in the analysis

$$\kappa = aP\bar{x}/T^{\frac{11}{2}}, \tag{2}$$

where

$$a = \frac{40}{\pi^4\sqrt{3}} \frac{e^2h^4}{cm(2\pi m)^{\frac{3}{2}}(k)^{\frac{11}{2}}} \frac{2{\cdot}43\chi_1^2}{\text{atomic mass}}$$

$$= 5{\cdot}62 \times 10^{19} \times \bar{\chi}^2/a, \tag{2'}$$

where $\bar{\chi}$ = is the ionization-potential in *electron-volts* ;
a = the atomic weight (not *mass*).

3. *Solution of the Problem.*—Let p' denote the radiation-pressure at depth h in the photospheric layers measured from some convenient reference level. A net flux πF traversing a layer ρdh, of absorption coefficient κ, communicates a momentum $\pi F\kappa\rho dh/c$. Hence

$$\frac{dp'}{dh} = \frac{\kappa\rho\pi F}{c}. \tag{3}$$

Milne has proved that the relation $p' = \frac{1}{3}aT^4$ holds to the extent to which $T^4 = \frac{1}{2}T_1^4(1 + \frac{3}{2}\tau)$ is an accurate solution of Schwarzschild's problem. The equation of mechanical equilibrium is

$$\frac{dp}{dh} + \frac{dp'}{dh} = g\rho. \tag{4}$$

Dividing (4) by (3),

$$1 + \frac{dp}{dp'} = \frac{cg}{\pi\kappa F}. \tag{5}$$

Let M be the mass of the star, L its bolometric magnitude ; then

$$g = GM/r_1^2 \quad \text{and} \quad \pi F = L/4\pi r_1^2, \quad i.e. \quad g/\pi F = 4\pi GM/L.$$

Further,

$$\kappa = \frac{aP\bar{x}}{T^{\frac{11}{2}}} = aP\bar{x}\left(\frac{\frac{1}{3}a}{p'}\right)^{\frac{11}{8}}, \tag{6}$$

where $\bar{x}$ is the average number of free electrons per atom in the photospheric layers, *i.e.*

$$\frac{P}{p} = \frac{\bar{x}}{1 + \bar{x}}. \tag{7}$$

Hence (5) becomes

$$1 + \frac{dp}{dp'} = \frac{4\pi cGM}{L} \frac{(1 + \bar{x})}{a\bar{x}^2(\frac{1}{3}a)^{\frac{11}{8}}} \frac{p'^{\frac{11}{8}}}{p}. \tag{8}$$

Make the substitution

$$p = u\frac{p'^{\frac{19}{16}}}{p_1'^{\frac{3}{16}}}, \tag{9}$$

where
$$p_1' = \tfrac{1}{3}aT_1^4. \tag{10}$$
(8) now becomes
$$\left(\frac{p_1'}{p'}\right)^{\frac{8}{16}} + p'\frac{du}{dp'} + \frac{19}{16}u = \frac{4\pi cGM}{L}\frac{(1+\bar{x})}{\bar{x}^2 a(\frac{1}{3}a)^{\frac{11}{8}}}p_1'^{\frac{3}{8}}\cdot\frac{1}{u}$$
$$= \left[\frac{4\pi cGM}{L}\frac{(1+\bar{x})}{\bar{x}^2 a\frac{1}{3}a}T_1^{\frac{3}{2}}\right]\cdot\frac{1}{u}$$
$$= \frac{A}{u} \text{ (say)}. \tag{11}$$

Now $(p_1'/p')^{\frac{8}{16}}$ deviates from unity only very little, and we replace it by unity. Hence we have
$$\frac{dp'}{p'} = \frac{udu}{A - u - \frac{19}{16}u^2}. \tag{12}$$
Write
$$A - u - \tfrac{19}{16}u^2 = -\tfrac{19}{16}(u-u_1)(u+u_2). \tag{13}$$
From (13) we see immediately that
$$\left.\begin{aligned} A &= (\tfrac{19}{16})u_1u_2\,; \\ \tfrac{16}{19} &= u_2 - u_1. \end{aligned}\right\} \tag{13'}$$
Integration of (12) gives us now
$$\log\frac{p'}{p_0'} = -\frac{\frac{16}{19}}{u_1+u_2}\left[u_1\log\left(1-\frac{u}{u_1}\right) + u_2\log\left(1+\frac{u}{u_2}\right)\right], \tag{14}$$
remembering the " boundary condition " that, when $u=0$,
$$p' = p_0' = \tfrac{1}{3}aT_0^4.$$
Since as the solution of Schwarzschild's problem we have
$$\frac{p'}{p_0'} = 1 + \tfrac{3}{2}\tau, \tag{15}$$
we see that as τ increases, p'/p_0' increases and u rapidly increases and approaches the limit u_1. Hence by (9) the ratio of gas-pressure to radiation-pressure increases slowly according to $u_1(p'/p_1')^{\frac{3}{16}}$. Thus
$$\frac{p}{p'} \text{ ultimately increases as } T^{\frac{3}{4}}. \tag{16}$$
It may be remarked here that the corresponding result on the Jeans-Milne analysis is that " p/p' ultimately increases as $T^{\frac{1}{4}}$." Now equations (14), (15) and (9) determine p', τ and p for a given u, p' in turn determining the temperature T. It now remains to determine the heights to which these values of u refer. In equation (3) substitute $\rho = p/(R/\mu)T$ and insert also the formula (6) for κ. We find
$$\frac{dp'}{dh} = \frac{\alpha\bar{x}^2}{1+\bar{x}}\left(\frac{\frac{1}{3}a}{p'}\right)^{\frac{11}{8}}\frac{p^2}{(R/\mu)T}\frac{gL}{4\pi cGM}, \tag{17}$$

or by (9),

$$\frac{1}{T}\frac{dT}{dh}=\frac{1}{4}\frac{1}{p'}\frac{dp'}{dh}=\frac{\frac{1}{4}\alpha\bar{x}^2}{1+\bar{x}}\frac{(\frac{1}{3}a)^{1.1}}{p'^{1.1}}\frac{p^2}{(R/\mu)T}\frac{gL}{4\pi cGM}=\frac{\frac{1}{4}g}{A(R/\mu)T}u^2\,; \qquad (17')$$

or by (13'),

$$\frac{dT}{dh}=\frac{\frac{1}{4}g}{R/\mu}\frac{u^2}{(\frac{19}{16})u_1u_2}. \qquad (18)$$

We observe that as $u\to u_1$,

$$\frac{dT}{dh}\to\frac{\frac{1}{4}g}{R/\mu}\frac{u_1}{(\frac{19}{16})u_2}, \qquad (19)$$

or again by (13'),

$$\left(\frac{dT}{dh}\right)_\infty=\frac{\frac{1}{4}g}{R/\mu}\frac{1}{(\frac{19}{16})+u_1^{-1}}. \qquad (19')$$

From (17') we get, after some minor reductions,

$$\frac{gdh}{(R/\mu)T}=-\frac{u_1u_2du}{u(u-u_1)(u+u_2)}. \qquad (20)$$

(20) *is identical* with Milne's equation (80) of his Bakerian Lecture. (Of course our u_1 and $-u_2$ are the roots of a slightly different quadratic equation in u and our variable u itself is differently defined.)

Denoting a mean value of T by $\overline{T}$ we find that (Bakerian Lecture, equation (81))

$$h-h_0=\frac{(R/\mu)\overline{T}}{g}\left[\log u-\frac{u_2}{u_1+u_2}\log\left(1-\frac{u}{u_1}\right)-\frac{u_1}{u_1+u_2}\log\left(1+\frac{u}{u_2}\right)\right]. \qquad (21)$$

We have now obtained the complete solution to our problem, with, of course, certain simplifying assumptions.

4. *Numerical Applications.*—We use the above formulæ to derive the physical characteristics of the photospheric layers of the " Sun," *i.e.* of a " model " whose mass, luminosity, surface gravity, T_1 etc., are the same as those of the Sun. Now Russell * estimates that the " level of ionization " in the solar atmosphere is such that the atoms of ionization-potential 8·3 volts are 50 per cent. ionized. Hence in our numerical work we take $\overline{\chi}=8{\cdot}3$ volts and $\bar{x}=\frac{1}{2}$. Also, to determine the value of α in (2), we need in addition a *mean* atomic weight. We assume this to be 1·5, to be in conformity with the estimates of Russell and Unsöld regarding the hydrogen-abundance in stellar atmospheres. Also, for " μ " in equation (18) we take 0·75 m_H. The results of the calculation are given below:

$$\alpha=2{\cdot}58\times10^{21}\,;$$
$$\bar{x}=\tfrac{1}{2}\,;$$
$$\mu=0{\cdot}75m_H\,;$$
$$\text{atomic weight}=1{\cdot}5\,;$$
$$\overline{\chi}=8{\cdot}3\text{ volts}\,;$$
$$g=2{\cdot}74\times10^4\text{ cm. sec.}^{-2}\,;$$
$$T_1=5740\text{ degrees.}$$

* H. N. Russell, *Ap. J.*, **70**, 11, 1929.

Using these values,

$$A = \frac{4\pi c G_1 M}{L} \frac{T_1^{\frac{3}{2}}}{\frac{1}{3}a\alpha} \frac{1+\bar{x}}{\bar{x}^2} = 6582 ;$$

$$u_1 = 74{\cdot}02 ;$$
$$u_2 = 74{\cdot}86 ;$$
$$u_1 + u_2 = 148{\cdot}88.$$

The physical structure of the photospheric layers calculated with the above data and the formulæ of the previous section is tabulated in Table I.

It is seen that the physical characteristics thus calculated offer a highly plausible description of the state of the Sun's photospheric layers, the one exception being that the pressures appear to be perhaps rather large. This is a consequence of the low absorption coefficient consequent on the 50 per cent. ionization assumed in the calculations. To illustrate the effect of ionization, therefore, we repeat the calculations with the following data:—

$$\bar{\chi} = 7{\cdot}812 \text{ volts} ;$$
$$\bar{x} = 1 ;$$
$$\mu = 0{\cdot}5 m_H ;$$
$$\text{atomic weight} = 1 ;$$
$$\alpha = 3{\cdot}43 \times 10^{21} ;$$
$$A = 1650 ;$$
$$u_1 = 36{\cdot}86 ;$$
$$u_2 = 37{\cdot}70 ;$$
$$u_1 + u_2 = 74{\cdot}56.$$

The results of the calculations are contained in Table II. We note the general reduction in pressure and the increased absorption coefficient.

TABLE I

u	Optical Thickness	Radiation-pressure, p', in Dynes cm.$^{-2}$	Gas-pressure, p, in Dynes cm.$^{-2}$	Temperature T Degrees	Temperature Gradient dT/dh Degrees Km.$^{-1}$	Depth h, Km.	Density ρ Gms. cm.$^{-3}$	Absorption Coefficient κ
0	0	1·386	0	4830	0	$-\infty$	0	0
10	0·0058	1·399	12·3	4841	0·0945	447·6	$2{\cdot}309 \times 10^{-11}$	28·43
20	0·0215	1·432	25·27	4868	0·3781	591·6	$4{\cdot}716 \times 10^{-11}$	57·07
30	0·0521	1·494	39·92	4921	0·8506	681·6	$7{\cdot}367 \times 10^{-11}$	86·08
40	0·1038	1·602	57·84	5008	1·513	764·5	$1{\cdot}049 \times 10^{-10}$	111·1
50	0·1933	1·788	82·38	5147	2·364	846·6	$1{\cdot}453 \times 10^{-10}$	134·8
60	0·3764	2·169	124·4	5402	3·404	935·2	$2{\cdot}090 \times 10^{-10}$	157·9
65	0·5688	2·568	164·6	5635	3·994	1034	$2{\cdot}654 \times 10^{-10}$	166·1
70	1·0410	3·550	260·4	6109	4·632	1185	$3{\cdot}872 \times 10^{-10}$	169·5
72	1·5930	4·699	373·7	6554	4·900	1315	$5{\cdot}178 \times 10^{-10}$	164·5
73	2·335	6·240	530·7	7036	5·037	1455	$6{\cdot}850 \times 10^{-10}$	157·9
73·5	3·334	8·320	752·1	7736	5·107	1631	$9{\cdot}016 \times 10^{-10}$	147·9
74	8·75	19·57	2095	9363	5·176	2137	$20{\cdot}32 \times 10^{-10}$	127·3

On comparing Tables I and II (particularly Table II) with table IX of Milne's Bakerian Lecture, it will be observed that there is practically no

difference between the *predicted* distribution of temperature, density, pressure, etc., in the photospheric layers as τ varies, and that deduced by Milne on the basis of his empirical law for the absorption coefficient. The physical structure of the photospheric layers as deduced by Milne has been generally regarded as representing what is "likely," and it is satisfactory that our results, based on a purely physical theory of the absorption coefficient, should be in general agreement.

TABLE II

u	τ	Radiation-pressure, p', in Dynes cm.$^{-2}$	Gas-pressure, p, in Dynes cm.$^{-2}$	Temperature T Degrees	Temperature Gradient dT/dh Degrees Km.$^{-1}$	Depth h, Km.	Density $\rho \times 10^{12}$ Gms. cm.$^{-3}$	Absorption Coefficient κ
0	0	1·386	0	4830	0	0	0	0
5	0·0052	1·397	6·146	4839	0·0629	468·2	7·693	57·15
10	0·213	1·432	12·64	4865	0·2514	607·5	15·71	113·2
15	0·522	1·494	19·96	4921	0·567	844·3	24·57	169·0
20	0·1037	1·602	28·93	5008	1·006	943·2	34·97	222·3
25	0·1937	1·789	41·20	5147	1·571	1065	48·48	268·6
30	0·3805	2·180	62·36	5409	2·263	1223	69·98	314·2
33	0·6380	2·712	89·15	5712	2·738	1374	94·51	332·7
35	1·0861	3·644	134·3	6137	3·080	1568	132·2	333·5
36	1·731	4·985	200·4	6651	3·26	1783	182·5	323·7
36·5	2·794	7·196	314·3	7292	3·35	2046	261·0	306·3
36·8	6·741	15·40	781·2	8819	3·40	2677	536·5	274·2

But our results differ in one point from those of Milne, in that *our linear depths* (in kms.) *are uniformly about* 40 *times larger*. This is due to the fact that we have assumed in our calculations a very low mean molecular weight, $0{\cdot}75m_H$ as against Milne's $20m_H$.

The predicted pressures and the absorption coefficients are of the expected order, though it should be remembered that this is largely connected with the low mean atomic weight used in our calculations.

5. *Alternative Treatment.*—It will be noticed that for the observed order of magnitudes for L, M, T, etc., A is very large—of order 1000. In fact we have

$$A = 1097\left(\frac{M}{M_\odot}\right)\left(\frac{L_\odot}{L}\right)\left(\frac{T_1}{5740}\right)^{\frac{3}{2}}\left(\frac{\alpha'}{\alpha}\right)\frac{1+\bar{x}}{\bar{x}^2}, \tag{22}$$

where α' is the coefficient in the expression (2′) for κ, calculated with $\bar{\chi} = 8{\cdot}3$ volts and $a = 1{\cdot}5$. From (13′) we easily obtain for u_1 and u_2,

$$u_1 = \sqrt{\beta^2 + 0{\cdot}18} - 0{\cdot}42\ ;$$

$$u_2 = \sqrt{\beta^2 + 0{\cdot}18} + 0{\cdot}42\ ;$$

where

$$\beta^2 = \tfrac{16}{19}A. \tag{22′}$$

Since β is very large we see that u_1 and u_2 are both practically equal to β. Hence we can rewrite (12) as

$$\frac{19}{16}\frac{dp'}{p'} = \frac{udu}{\beta^2 - u^2}. \tag{22″}$$

(22″) integrates immediately, and we obtain, remembering that when $u=0$, $p'=p_0'$,

$$p'=p_0'\left(1-\frac{u^2}{\beta^2}\right)^{-\frac{5}{19}} \tag{23}$$

By (9) we have for the gas-pressure,*

$$p=up_0'\left(1-\frac{u^2}{\beta^2}\right)^{-\frac{1}{2}}2^{-\frac{3}{16}}. \tag{23'}$$

In obtaining (23′) we have used the relation, $p_0'=\frac{1}{2}p_1'$.

To determine the temperature gradient we proceed as before, and by (17′) we have

$$\frac{dT}{dh}=\frac{\frac{1}{4}g}{A(R/\mu)}u^2. \tag{24}$$

By (23), on the other hand,

$$1-\frac{u^2}{\beta^2}=\left(\frac{T_0}{T}\right)^{\frac{19}{2}}; \tag{25}$$

or

$$u^2=\beta^2\left[1-\left(\frac{T_0}{T}\right)^{\frac{19}{2}}\right]. \tag{25'}$$

Inserting (25′) in (24) and using the relation (22′) we obtain

$$\frac{dT}{dh}=\frac{4}{19}\frac{g}{R/\mu}\left[1-\left(\frac{T_0}{T}\right)^{\frac{19}{2}}\right]. \tag{26}$$

Now as we descend into the photospheric layers the second term in brackets in the above equation very soon becomes negligibly small because of the high power to which a quantity less than unity is raised. Thus, if $T=1{\cdot}18T_0$, $(T_0/T)^{\frac{19}{2}}=0{\cdot}2$, and if $T=1{\cdot}275T_0$, $(T_0/T)^{\frac{19}{2}}=0{\cdot}1$. Hence we have, to a good approximation, for layers not immediately near the surface and for which $T\geqslant 1{\cdot}25T_0$,

$$\frac{dT}{dh}=\frac{4}{19}\frac{g}{R/\mu}, \tag{27}$$

or

$$h-h_0=\frac{19}{4}\frac{R/\mu}{g}T. \tag{28}$$

Comparing (27) with (19′), and remembering that $u_1^{-1}<<1$, we find that what we have called previously "$\left(\frac{dT}{dh}\right)_\infty$" is in fact a very good approximation for layers not immediately near the boundary. Equation (26) in fact shows the extent to which we have to descend in the photospheric layers

* From (23) and (23′) we have

$$p/p'\propto u\left(1-\frac{u^2}{\beta^2}\right)^{-\frac{3}{5}}\propto up'^{\frac{3}{16}}\propto uT^{\frac{3}{4}}.$$

But as T increases, $u\to\beta$ and is practically constant. Hence that "p/p' ultimately varies as $T^{\frac{3}{4}}$" comes out from this treatment also.

before our approximation becomes valid. The temperature will have to increase by about 25 per cent. over the boundary temperature. In fig. 1 the results of Tables I and II are plotted—the temperature against the depth —and we see at once how soon the linear relation between h and T sets in—

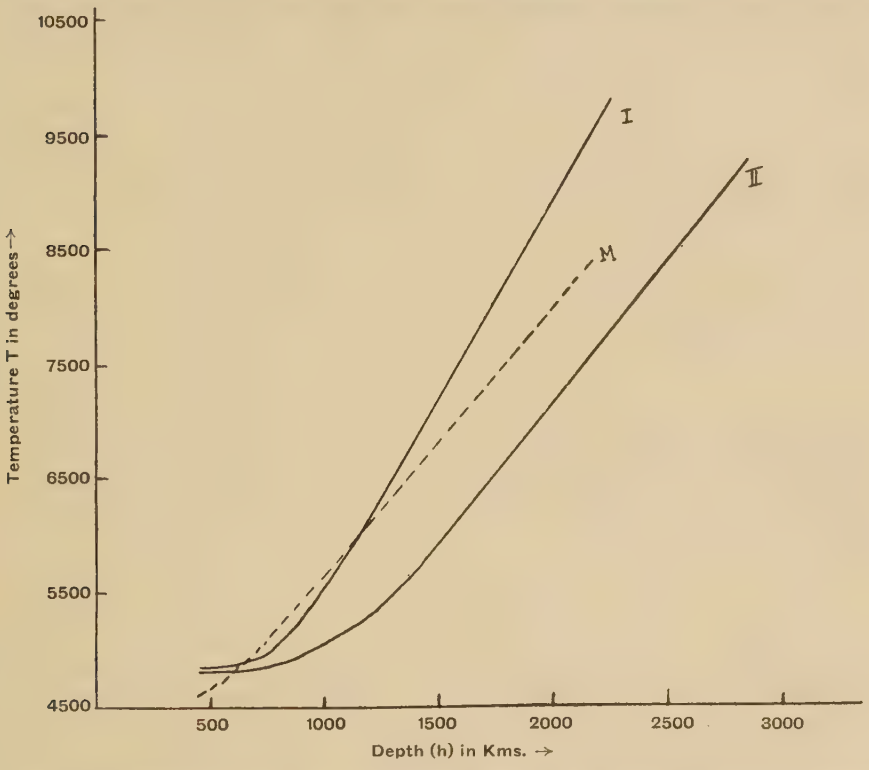

FIG. 1.—*Physical structure of the photospheric layers of the "Sun." The plot of temperature against depth to exemplify the linear relation between the temperature* (T) *and the linear depth* (h) *for layers not immediately near the boundary.*

I—Corresponds to the data of Table I of this paper.
II—Corresponds to the data of Table II of this paper.
M—Corresponds to the data of Table X of Milne's Bakerian Lecture.

which equations (26) and (27) jointly predict. (The dotted curve is the one obtained by using Milne's tabulated values. The linear relation is exemplified here also.*)

Again, (28) can be rewritten, with the help of (23), in the form

$$h - h_0 = \frac{19}{4}\frac{(R/\mu)}{g} T_0 \left(1 - \frac{u^2}{\beta^2}\right)^{-\frac{2}{19}}. \qquad (29)$$

* If $\kappa \propto P/T^{\frac{9}{2}}$ then the relation (26) will be replaced by

$$\frac{dT}{dh} = \frac{4}{17}\frac{g}{(R/\mu)}\left[1 - \left(\frac{T_0}{T}\right)^{\frac{17}{2}}\right], \qquad (26')$$

and instead of (28) we have

$$h - h_0 = \frac{17}{4}\frac{(R/\mu)}{g} T. \qquad (28')$$

Equation (29) is of course true only after we have descended some distance into the photospheric layers.

The solution for h for layers near the boundary ($T \leqslant 1{\cdot}25T_0$) can be obtained by a procedure analogous to that employed in § 3. Now since T deviates only a little (less than 25 per cent.) from the boundary value, the use of a " mean " temperature $\overline{T}$ is amply justified. The solution is (*cf.* equation (21))

$$h - h_0 = \frac{(R/\mu)\overline{T}}{g}\left[\log u - \tfrac{1}{2}\log\left(1 - \frac{u_2}{\beta^2}\right)\right], \tag{30}$$

where $\overline{T}$, as before, represents some mean value of T for the layers traversed. (30) can be rewritten in a more convenient form. By (23′),

$$h - h_0 = \frac{(R/\mu)\overline{T}}{g}\left[\log\left\{\frac{p}{p_0'} \cdot 2^{\frac{3}{16}}\right\}\right], \tag{31}$$

or

$$h - h_0 = \frac{(R/\mu)\overline{T}}{g}[\log p - \log p_0' + 0{\cdot}130]. \tag{32}$$

Equations (23), (23′), (32), and (29) give the complete solution to our problem.

From (28) we can obtain one other useful piece of information. If we consider two stars with the same μ and boundary temperature (or T_1), then, since

$$\frac{p}{p_0'} = 1 + \tfrac{3}{2}\tau,$$

it is clear that for both these stars equality of τ signifies equality of p' and therefore of T at the corresponding points. Then relation (28) shows that if we neglect the arbitrary additive constant, the h's for the two stars should be inversely as the surface gravities, or, to put it more precisely, for *two stellar atmospheres of the same molecular weight* μ *and boundary temperature* T_0, *the thicknesses of the photospheric layers between places of equal temperature* T' *and* T'' ($T' < T''$) *in the two stars are inversely proportional to the surface gravities if* $T' > 1{\cdot}25\ T_0$.

It is clear that even if T' is not greater than $1{\cdot}25\ T_0$, the proportionality of the " thicknesses " to g^{-1} will still be approximately valid if T'' is sufficiently large. Thus if (following Milne) we adopt $\tau = 4$ * as defining the greatest depth to which we can see at the centre of the disc, it follows that the thicknesses between $\tau = 0{\cdot}01$ (say) and $\tau = 4$ should be roughly proportional to g^{-1}. This result has previously been noted by Milne from his numerical calculations contained in his Bakerian Lecture, but the above gives the result a precise analytical form and shows that the proportionality $(h'' - h') \propto g^{-1}$ should strictly hold good if the boundary temperatures are the same and the temperature T' (corresponding to h', the upper level of the two) is appreciably greater than T_0.

6. *General Remarks.*—It will be noted that the above calculations are not

* Actually $\tau = 4$ corresponds to $T'' = 1{\cdot}626\ T_0$ and $\left(\frac{T_0}{T}\right)^{\frac{1}{2}} = 0{\cdot}01$.

strictly deduced from the physical theory alone. Thus we have used in our numerical calculations values for $\bar{x}$ near that deduced by Russell for the level of ionization from different considerations. This is not, however, an intrinsic defect in the method, and is essentially due to the fact that to secure simplicity in the analysis we had to introduce a mean value $\bar{x}$ for the degree of ionization instead of treating this as a variable as we ought to have done ; and the penalty to be paid for this is that we have to submit to our method still having a semi-empirical air about it. For, the fact that we assume *beforehand* a mean value for $\bar{x}$ presupposes that we have already a knowledge of this from other sources. Thus for temperatures of the order of 6000°, if we take $\chi = 13{\cdot}54$ volts (say) then it is clear *a priori* that $\bar{x}$ can only be too small and the results of the calculations cannot surely be regarded as " representative." For, if $\bar{x} \to 0$ then $\kappa \to 0$, $A \to \infty$, and $dT/dh \to 0$, and the photospheric layers tend to have an infinite extension. Hence to be " reasonable " we must have values of x not too small and which should not be inconsistent with our choice of $\bar{\chi}$ and the physical conditions—a knowledge of which is presupposed when we assume $\bar{x}$.*

7. *Summary.*—(1) The physical structure of the photospheric layers is studied on the basis of an accurate theory of the absorption coefficient.

(2) It is found that ultimately the ratio of the gas-pressure to the radiation-pressure increases as $T^{\frac{3}{4}}$.

(3) Numerical applications are made to two typical cases, and though the values are derived purely on the basis of the physical theory and certain plausible assumptions regarding the " level of ionization," there seems nothing " impossible " about the predicted physical characteristics of the photospheric layers. The pressures and the absorption coefficients are of the expected order.

(4) An alternative treatment of the photospheric problem is given which shows that the linear depths are proportional to the temperatures for layers not immediately near the boundary.

(5) It is also pointed out that for two stars with the same μ and T_0 at points of equal τ (not too small) the linear depths are inversely as the surface gravities.

In conclusion I wish to record my thanks to Professor E. A. Milne, at whose suggestion the calculations presented here were undertaken.

* The consistency of the calculations can be tested as follows : To start the calculation we assume a mean value for $\bar{x}$. Then the method of the paper provides a complete march of T and p down the photospheric layers, and with these calculated values of T and p we could calculate x at each level. The mean of these x's should be equal to the initial value of $\bar{x}$ assumed. If it is different the calculation should be repeated with a new starting value of $\bar{x}$.

A PHOTO-ELECTRIC MICROPHOTOMETER USING VALVE AMPLIFICATION.

C. S. Beals.

The use of the photo-electric cell as a sensitive element for the measurement of the densities of blackening of photographic plates has become general of recent years, and a number of microphotometers have been built by different investigators in which the photo-electric cell is employed. These include the now well-known Zeiss instrument, the Koch and Goos * instrument, and the new microphotometer of the Cambridge Solar Physics Observatory.† These instruments are all of the self-registering type, and use electrostatic methods of recording.

The purpose of the present article is to describe an instrument recently built in the physical laboratory of the Dominion Astrophysical Observatory, which is of the direct-reading type and employs as a recording system a thermionic valve amplifier in conjunction with a galvanometer and scale. A limited amount of experience in the photometry of stellar spectra had suggested to the writer that for certain types of work a flexible and sensitive direct-reading instrument would be more convenient than one of the self-registering type. This idea was confirmed by correspondence with Dr. G. R. Harrison, who, with Hesthal,‡ had constructed a direct-reading photo-electric microphotometer for the purpose of measuring line intensities in laboratory spectra. Accordingly, an attempt was made to design an instrument which, it was hoped, would be specially suited for the investigation of certain astrophysical problems being studied at this observatory. The necessary requirements were as follows :—

(1) Convenience and simplicity of operation so that limited regions of the spectrum could be studied without elaborate preparation.
(2) Considerable sensitivity so that both low and high densities could be accurately measured.
(3) High resolving power for the measurement of narrow lines.
(4) Means for the precise determination of the wave-length of that part of the spectrum whose density was being measured.

The completed instrument is described in this paper. It is similar in principle to the microphotometer built by Hesthal and Harrison in which a valve amplifier with a galvanometer and scale was employed. It has proved to be reasonably inexpensive and easy to construct, and has more than fulfilled expectations in its performance. The writer is very much indebted to Dr. G. R. Harrison for a discussion of the physical principles

* *Zeits. f. Phys.*, 44, 855, 1927.
† *M.N.*, 91, 191, 1930.
‡ *Phys. Rev.*, 34, 543, 1929 (abstract).

involved in the construction of a microphotometer of this type. Thanks are also due to Dr. H. D. Arnold, Director of the Bell Telephone Research Laboratories, and Mr. P. Higgs, of the University of Washington, for suggestions relating to the use of a valve amplifier.

The Electrical System

Descriptions of amplifying systems similar to the one used here have been published by Brentano,* by Wynn-Williams,† and more recently by

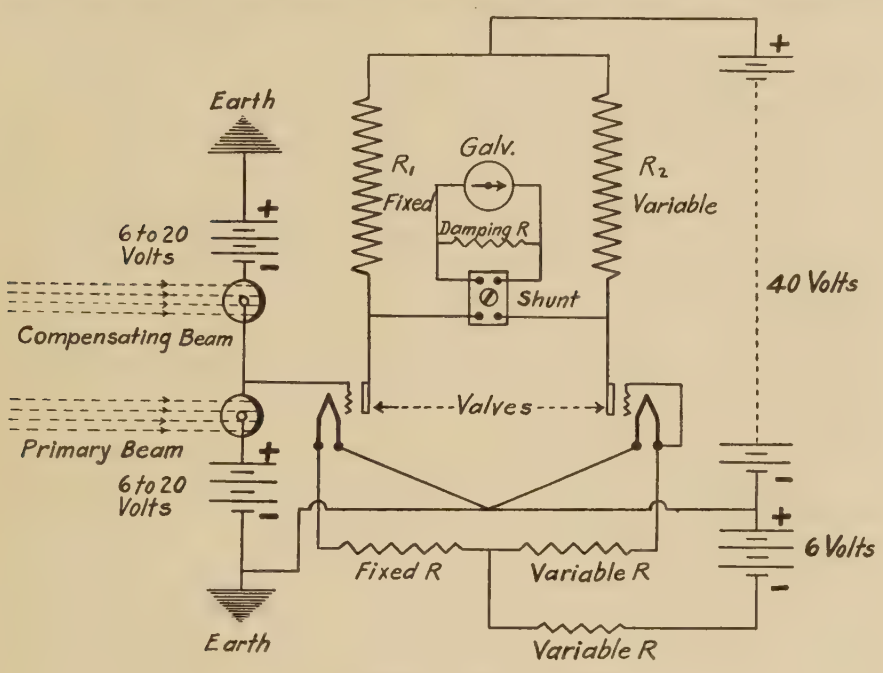

FIG. 1.—*Diagram of Electrical System.*

Nottingham.‡ Nottingham has discussed in some detail the physical characteristics of such an amplifier, so that only a bare description is given here. A diagram of the electrical parts is shown in fig. 1. Two thermionic valves form two of the arms of a Wheatstone bridge, while two resistances of from 8000 to 10,000 ohms form the other two arms. A galvanometer with shunt and damping coil is connected as shown in the diagram. One of the valves is made into a diode by connecting the grid and filament, while the grid of the other valve is connected to the photo-electric cells. A current flowing to the grid of the amplifying valve unbalances the bridge and causes a deflection of the galvanometer corresponding to an amplification factor of the order of 5000. Since the current is proportional to the light absorbed by the photo-electric cell, the galvanometer deflections provide

* *Nature*, **106**, 532, 1921.
† *Phil. Mag.*, **6**, 324, 1928.
‡ *Journ. Franklin Inst.*, **209**, 287, 1930.

a measure of the intensity of the light, and, indirectly, of the density of the photographic plate or other absorber placed in the path of the beam.

The galvanometer is a Leeds and Northrup F4447 type B, and as supplied by the makers had a period of 6 seconds, a coil resistance of 500 ohms, and required 10,000 ohms external damping. Its characteristics have been changed by installing a suspension of period $1\frac{1}{2}$ seconds. An Ayrton shunt of total resistance 10,000 ohms is used to vary the sensitivity, and an external damping coil of 2000 ohms provides practically complete damping.

The bridge resistances have each a total resistance of 10,000 ohms. One of them is an ordinary box-resistance with plugs for varying the total resistance in the circuit. The other is a variable resistance in a box with four dials having steps of 1, 10, 100 and 1000 ohms.

The valves used are of type Ux 201 A, with a plate impedance of the order of 10,000 ohms. They are used with a plate potential of 40 volts and a filament current of 0·15 amp. approximately. The filaments are lighted by a 6-volt automobile battery. A variable resistance is used for controlling the total current through the filaments, while two additional resistances, one fixed and one variable, make it possible to compensate for differences in valve characteristics by altering the current through the filament of one valve relative to that of the other. It is probable that almost any well-built radio valve could be used successfully in such an amplifier.

The photo-electric cells are Visitron type A.V. gas-filled cells manufactured by the G.M. Laboratories, Chicago. These cells are of the cæsium on silver type, and are highly sensitive in the visible and near infra-red region. They are operated on potentials of from 6 to 20 volts. Since these voltages are too low to produce the ionization currents on which the sensitiveness of a gas-filled cell depends, it is probable that vacuum cells would serve the purpose equally well. The cells are connected to the amplifier in the manner originally described by Koch,* which is illustrated in the diagram. The light from the lamp which provides the illumination for producing the photo-electric current is divided into two beams. The first beam passes through the photographic plate whose density is to be measured and falls on the first cell. The second beam, whose intensity can be varied by means of a diaphragm, follows a different path and by a series of reflections is caused to fall on the second cell. The second cell performs the function of a variable grid leak, and can be used to alter the sensitivity or to provide partial compensation for changes in the intensity of the source of light.

All parts of the electrical system, including batteries, resistances, valves, photo-electric cells and galvanometer, are enclosed in a single large sheet-metal box, which provides complete shielding from external electromagnetic disturbances. Control dials and switches are mounted in convenient positions in front of the box so that all necessary adjustments, such as balancing the bridge, altering the sensitivity, changing the zero point of the galvanometer, etc., can be performed without lifting the cover.

* *Ann. d. Phys.*, **30**, 705, 1912.

The Optical System

The optical system is illustrated in fig. 2. An ordinary 30 c.p. automobile headlight is used as a source of light. It is fed by two large 6-volt storage batteries in parallel, and is satisfactorily steady if run for half an hour before the instrument is used. In front of the lamp is a glass condenser, consisting of two simple plano-convex lenses mounted with their convex faces inward as in a projection lantern. A converging beam from the condenser falls on the photographic plate, which is held in a micrometer carriage having two screw motions at right angles to one another. The plate is slightly behind the true focus of the condenser, so that a bright out-of-focus patch instead of a sharp image of the filament appears on the plate. The projector is a Dallmeyer cinematograph lens of F 1·5 aperture and 16 mm. focal length. It is set up in a micrometer-focussing mount, and has an extension consisting of a brass tube which enables the lens to be brought close to the surface of the plate and yet leaves room for convenient manipulations in focussing.

An image of the spectrum enlarged 17 times is projected on the screen, whose dimensions are 8×10 inches. In the centre of the screen is an accurately machined slit, variable both in length and in width. The variation in length is provided by the direct vertical motion of two celluloid strips above and below the slit. These strips have etched on their surfaces fine black lines which are continuous with the slit itself. These black lines serve the same purpose as the cross-hairs in an ordinary micrometer-measuring machine. With their aid it is possible to make accurate settings on the comparison lines on either side of the star spectrum and so to determine precisely the wave-length of the region of the spectrum under examination. The actual length of slit used varies with the width of the spectrum. It is usually about 5 mm.

The width of the slit, which opens symmetrically, is controlled by a micrometer screw, and slit-widths can readily be estimated to 0·01 mm. Since the magnification is approximately 17, a slit-width of 0·17 mm. corresponds to a distance on the plate of 0·01 mm. Owing to plate grain, smaller distances than this on the plate are usually of little significance, so that slit-widths of less than 0·1 mm. are seldom used. The light, after passing through the slit, falls on the photo-electric cell, which is 36 inches distant No lens is used between the slit and cell, as the diverging beam from the slit forms a patch of light on the cell which is slightly smaller than the sensitive surface.

The path of the beam to the compensating cell is clearly shown in the diagram. Three mirrors are used, and a diaphragm enables the intensity of the compensating beam to be varied at will by the operator.

Method of Recording

The galvanometer is provided with a plane mirror and a lens of 1 metre focal length. For the purpose of making readings a semi-transparent celluloid scale is used. The scale is 60 cm. long, has millimetre divisions,

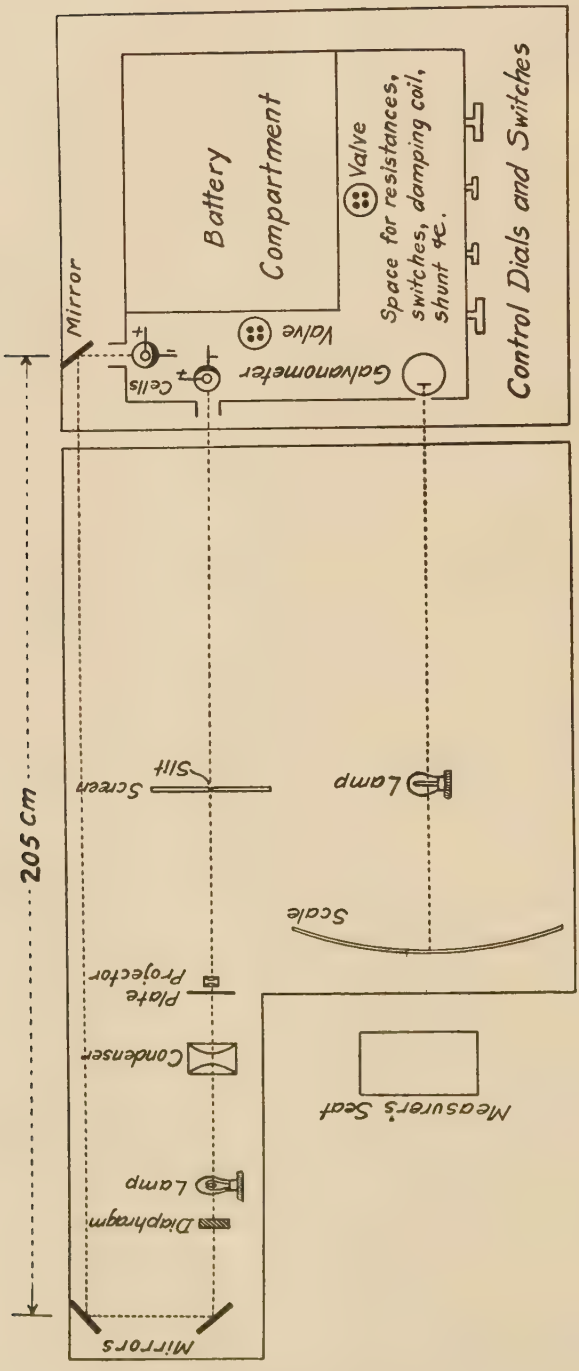

FIG. 2.—*Diagram of Microphotometer.*

and is used at a distance of 120 cm. from the galvanometer. The use of a lens necessitates a considerably curved scale, and somewhat better results might be obtained with a concave mirror. Contrary to the experience of Harrison and others, the writer has found the image of a vertical lamp filament by far the best and most accurate index to read. To make the scale more easily visible, a strip of white drawing paper has been placed behind and in contact with the etched figures and divisions. The scale is then illuminated from the front and is brought into high relief by the background of white paper. The image of the lamp filament appears below the scale, in the manner illustrated in fig. 3. The scale is placed directly in front of the operator, about 18 inches above the table, and in such a position that the eye can easily focus on the index and scale divisions. Fractions of millimetres can easily be estimated, but in practice readings are

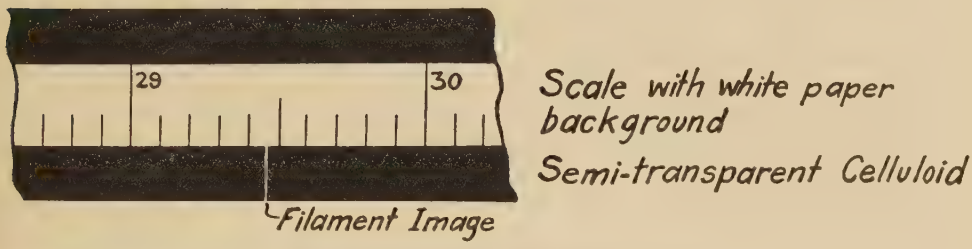

Fig. 3.—*Method of Using Lamp and Scale.*

made to the nearest millimetre, the sensitivity being so adjusted that the range of density under measurement corresponds to a distance on the scale of about 300 mm.

The Instrument in Use

The instrument has proved to be very convenient in use, and has given comparatively little trouble. The only serious difficulties so far encountered have been due to insufficiently charged batteries and faulty soldered joints in the amplifier circuit. It is unnecessary to darken the room in which the instrument is used, and no difficulty has been encountered from electromagnetic disturbances. Some of the more important points concerning its performance may be outlined as follows :—

Steadiness of Zero.—In order to use successfully an amplifier of the type described here, it is necessary so to adjust the amount of light falling on the photo-electric cell and the sensitivity of the galvanometer that quick, irregular variations in the zero are inappreciable as compared with the magnitude of the galvanometer deflection being measured. These conditions have been attained in the present instrument by using the galvanometer previously described and reducing its sensitivity to 1/10 by means of the Ayrton shunt. With this combination irregular variations are reduced to a range of $\pm 0{\cdot}5$ mm. With sensitivities ordinarily used, irregular variations in the photographic emulsion cause deviations of $\pm 10{\cdot}0$ mm. in the readings, so that minute irregularities in the zero are of no significance.

A steady slow creep of the zero is generally observed during a run. It is

usually small, and can be allowed for by checking from time to time during the course of a set of readings. For most purposes it is sufficient to check the zero at the beginning and end of a run and compute the corrections to be applied on the assumption that the creep is uniform. Since the instrument has been put in use, some 55 runs have been made of about 200 readings per run. The average creep has been 1·8 per cent. of the maximum deflection. Of the 55 runs, in 34 the creep has been 1 per cent. or less ; in 13 it has been between 1 and 2 per cent. ; in 5 it has been between 2 and 5 per cent. ; while in 3 it has been over 5 per cent. If the creep is less than 5 per cent. and if the corrections are carefully applied, it is believed that no error of importance is introduced. For precise work a run showing a creep of more than 5 per cent. is discarded.

Sensitivity.—The sensitivity of the instrument can be varied within wide limits, and can be made sufficient to cause a deflection of 50 mm. or more for the smallest difference in density that is clearly perceptible to the eye. The following means of altering the sensitivity are available :—

(1) Altering the intensity of the illumination on the compensating cell.
(2) Altering the slit width.
(3) Altering the voltage on the photo-electric cells.
(4) Altering the intensity of the source of light by changing the current through the lamp.
(5) Changing the sensitivity of the galvanometer by means of the Ayrton shunt.
(6) Changing the suspension of the galvanometer.

Usually only (1) or (2) is used, though for special work it may be necessary to employ any of the methods enumerated for altering the characteristics of the instrument.

The instrument can be adjusted for the accurate measurement of very high or very low densities. For the measurement of a plate showing very high contrast its accuracy is limited only by the length of its scale, which is 600 mm.

Resolving Power.—It is difficult to make quantitative tests of the resolving power of a microphotometer owing to the impossibility of securing a test object which reproduces the conditions encountered in measuring a photographic plate. The resolving power of the present instrument, however, appears to be considerably greater than that of the photographic plate, and when a very fine slit is used, deviations in the readings, due to the grain of the plate, become troublesome. In order to get even a moderately smooth record it is necessary to use a slit width of 0·2 mm., or greater. In order to facilitate comparison with other instruments, the diffraction pattern of the edge of a razor blade placed in the position of the photographic plate is shown in fig. 4. The image of the edge was brought into sharp focus on the screen, and the micrometer readings and galvanometer deflections were recorded in the same manner as in measures of photographic density. A slit width of 0·05 mm. was used.

Speed.—The time required to place a plate in position and make a run of

300 readings, which include the measurement of a strip of the spectrum and of a calibration strip, is about 45 to 60 minutes. Fortunately, the characteristics of the instrument are such that a rapid run increases rather than decreases the accuracy. As a consequence a comparatively large number of measures

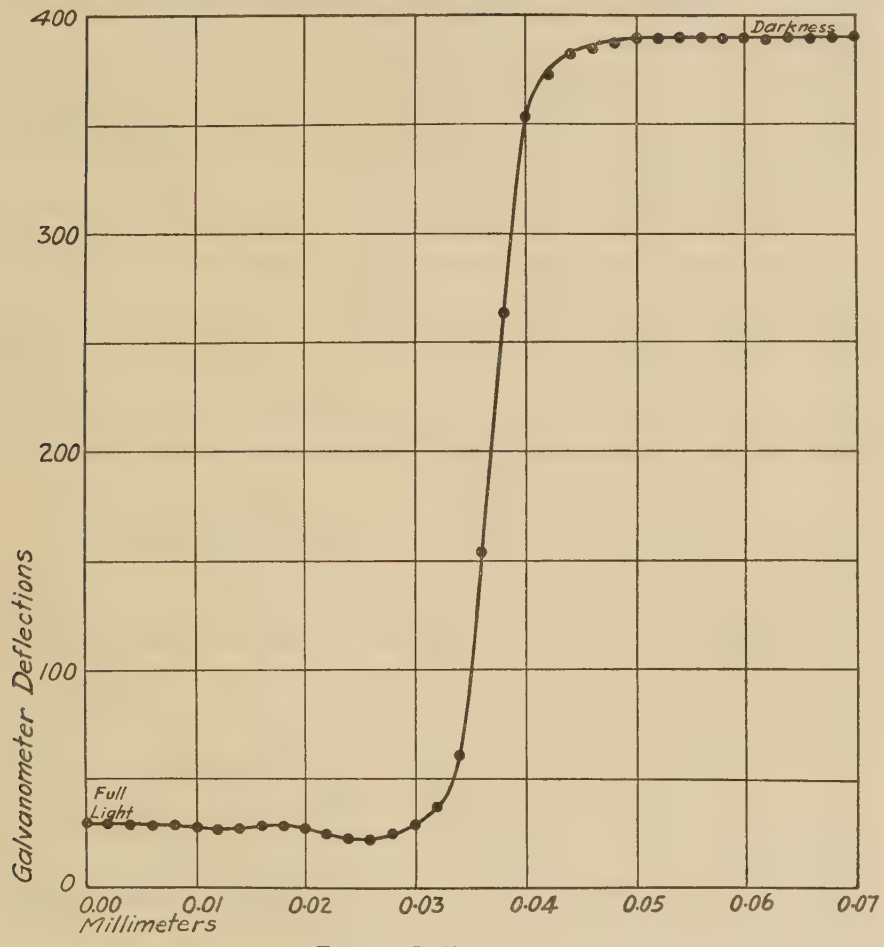

Fig. 4.—*Knife Edge Test.*

can be made in a short time, and the time taken in actually making the measures becomes small in comparison with that usually taken in reduction.

Examples of Actual Measurement

1. *Wolf-Rayet Band.*—Fig. 5 represents the contour of the emission band λ 4686 in the star H.D. 192163. Intensities are given in relative units and have been referred to the top of the band considered as 10. Measures of density have been translated into relative intensities by means of the usual characteristic curves. These curves have been plotted from measures of calibration strips impressed on the plates by means of a neutral tint wedge used with an auxiliary spectrograph. The exposure times of the calibration

strips were made as closely as possible equal to the exposure of the star spectrum.

Six of the plates cover the range of intensity from the top of the band to intensity 3·5 approximately. One plate covers the entire intensity range from the top of the band to the extreme edges, while two plates cover the range from approximately 4·5 to the edges of the band.

The plates used were mainly E33 and Barnet, though one spectrogram was taken with an Ilford Empress plate. High contrast developers were used, one due to Ross and the other to Parkhurst.* The E33 plates were very satisfactory in their fineness of grain, but were too slow to record the

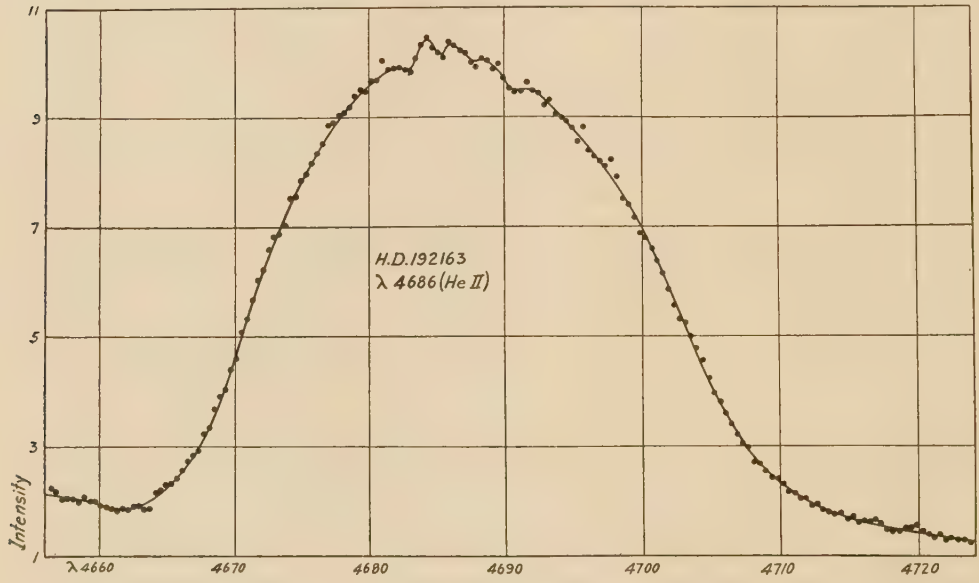

FIG. 5.—*Contour of Wolf-Rayet Emission Band.*

extreme edges of the band. The Barnet plates (H and D 1500) gave reasonably satisfactory results in spite of a rather coarse grain, and their superior speed was a factor of great importance in recording the faint continuous spectrum on either side of the band. The Ilford Empress plates are unsurpassed for evenness and fineness of grain, but are too slow to be very practicable for any but rather bright objects. The upper part of the graph from the top of the band to intensity 3·5 has been plotted from measures of seven plates, and each point, on the average, is the mean of seven observations. Since the measures were satisfactorily accordant, a reasonable amount of confidence in the result seems justified. Below 3·5, measures are from three plates only, and each point is a mean of three observations. The three plates showed excellent agreement, however, and it is hoped that there are no serious errors. In making the measures the feature of the instrument that permits precise measurement of the wave-length of each point relative to the comparison spectrum has proved of very great convenience. It makes unnecessary the arbitrary process of "fitting" contours

* *Pub. Dom. Obs.*, **8**, 121, 1924.

derived from separate plates, and not only adds to the accuracy but makes reduction much less laborious.

Theoretical predictions concerning the forms of Wolf-Rayet bands, as well as a number of actual measures, have already been published by the writer.* The band contour shown in fig. 5, however, brings into prominence two aspects not clearly shown in any previous measures.

The first of these is the appearance of complicated structural features on the top of the band, of a character somewhat similar to the structure that has been observed in bands in the spectra of novæ. This observation may be of considerable theoretical importance, since it suggests an additional indication of the fundamental similarity which the writer has endeavoured to establish between the spectra of novæ and Wolf-Rayet stars.

The second point, and one that is also of some theoretical importance, is the definitely asymmetrical character of the band as shown in the graph of fig. 5. According to the theory of Wolf-Rayet emission developed by the writer,† absorption would be expected on the violet edges of emission bands. In some cases such absorption has been observed to take the form of diffuse absorption lines bordering the violet edges of bands. It might be expected also that the absorption would act to add to the steepness of the contour on the violet edge, producing an asymmetry similar to that which is shown in fig. 5.

2. *Interstellar Calcium Lines.*—The graphs of fig. 6 show contours obtained for the two interstellar calcium lines H and K as well as the hydrogen line $H\epsilon$ in the spectrum of the star 9 Camelopardalis (H.D. 30614). In the case of the two interstellar lines the forms of the contours are almost undoubtedly due mainly to the properties of the spectrograph, and they have been determined for the purpose of obtaining accurate values of the total integrated intensities relative to the continuous spectrum. Three plates have been measured, and each point on the graph represents a single observation. In measuring such narrow lines it is necessary to use a narrow slit on the microphotometer. This greatly increases irregularities due to plate grain, and the scatter of the points on the diagram is due almost entirely to this cause.

The points due to different plates are clearly indicated on the diagram, and in spite of the relatively large scatter the agreement between plates is reasonably satisfactory. In determining the total integrated intensity of calcium H it is necessary to guess at the form of the violet wing of the line $H\epsilon$. It is naturally difficult to be certain of the magnitude of the errors introduced by an extrapolation of this sort. In this connection it is interesting to compare the value of the intensity ratio K/H with a value of the same quantity for this star determined by Struve and Elvey.‡ The actual quantities involved are represented by the integral

$$\int_{\nu_1}^{\nu_2} \frac{\partial I}{\partial x} \frac{\partial x}{\partial \nu} d\nu.$$

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* *M.N.*, **91**, 966, 1931.

† *Ibid.*, **90**, 202, 1929 ; *Pub. D.A.O.*, **4**, No. 17, 1930.

‡ *Zeits. f. Astrophysik*, **1**, 314, 1930.

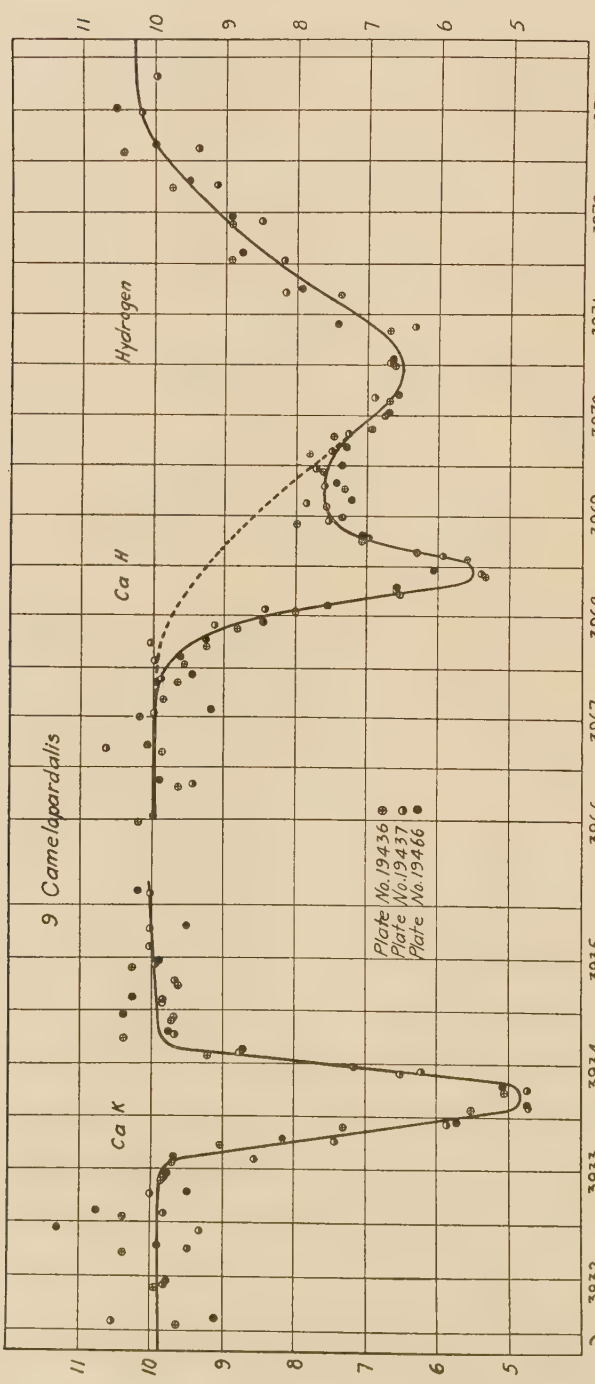

FIG. 6.—*Contours of Interstellar Calcium Lines and* H_ϵ.

The quantity $\frac{\partial I}{\partial x}$ represents the observed intensity, while $\frac{\partial x}{\partial v}$ depends on the dispersion of the spectrograph. The integration is performed mechanically, the limits of integration being the extreme edges of the wings of the line. In the case of the Ca^+ H line, the value of the integral represents the area enclosed between the full line and the dotted line in fig. 6.

Struve and Elvey's value for the ratio K/H is 1·27. The value of K/H in the present determination is also 1·27. Apparently Struve and Elvey's value is the result of measures from a single plate. If this is so, the precise agreement of the two determinations is probably accidental. The agreement is encouraging, however, as suggesting that reliable values of the total integrated intensities of these narrow lines may be derived from small dispersion spectrograms. The dispersion used by Struve and Elvey was 17 A. per mm., while that used by the present writer was 33 A. per mm., approximately.

The high resolving power and sensitivity of the new microphotometer make it particularly suitable for measurements of this sort, and the writer hopes in the near future to undertake a considerable programme of work involving problems connected with interstellar lines.

The writer is indebted to Mr. S. N. Hill, of the staff of this observatory, for assistance in the measurements and computations of this paper. The drawings that have been reproduced have also been done by Mr. Hill.

Dominion Astrophysical Observatory,
Victoria, B.C., Canada :
1931 *December* 1.

PHOTOGRAPHIC OBSERVATIONS OF DOUBLE STARS.

(Second List.)

W. M. Smart, M.A., D.Sc.

(Communicated by the Director, Cambridge Observatory)

This list of measures of double stars completes the survey of all the photographic plates taken with the Sheepshanks telescope up to the end of 1930. As in the first list,* the right ascensions and declinations of the stars are given for 1930·0. Position angles have been corrected for the positions of the images on the plates and for the effect of precession between the epoch of observation and 1930. In the second column is the designation of the star when known, the numbers referring either to Burnham's General Catalogue (β.G.C.) or to the Bonner Durchmusterung (B.D.). The accuracy of the measures is of the same order as in the first list. All the measures were made by Mr. H. E. Green, M.A.

* *M.N.*, 92, 37, 1931.

List of Double Stars

No.	Star (β.G.C. or B.D.)	R.A. (1930)	Dec. (1930)	Epoch 1900 +	Position Angle	Distance	Magnitudes	No. of Observations
		h m	° ′		°	″		
1	58° 2699	0 2·8	58 37	4·55	261·8	13·43	10·0 : 10·5	5
				26·87	260·7	13·55	...	6
2	58° 1	0 4·6	58 51	26·87	356·8	6·93	9·0 : 11·5	4
3	...	0 4·7	59 11	26·87	33·6	9·14	11·0 : 11·5	3
4	27	0 5·4	59 14	4·55	324·6	23·50	8·8 : 9·0	8
				26·87	325·2	24·00	...	6
5	28	0 5·5	44 21	22·86	77·1	15·80	8·5 : 10·5	2
6	57° 12	0 6·3	58 24	26·87	312·4	14·42	9·5 : 12·0	6
7	52	0 7·8	59 23	4·55	236·5	7·61	7·5 : 11·0	8
				26·87	235·2	7·63	...	6
8	43° 35	0 12·0	44 3	4·98	160·5	10·13	10·0 : 10·5	7
				23·50	160·6	9·98	...	8
9	99	0 12·7	43 13	4·63	75·8	9·28	6·0 : 10·5	12
				23·50	76·0	9·44	...	10
10	43° 40	0 13·1	44 4	4·63	84·9	8·30	9·1 : 10·0	12
				23·50	80·9	8·06	...	10
11	131	0 15·0	43 25	4·50	240·1	6·17	6·5 : 10·0	8
12	42° 61	0 17·2	43 20	23·20	96·3	8·90	8·5 : 11·5	3
13	29° 86	0 25·7	30 13	22·87	193·1	6·89	9·0 : 9·5	7
14	...	0 39·0	44 55	21·80	173·3	9·02	11·5 : 12·0	3
15	...	0 40·7	44 53	21·80	332·2	2·48	10·5 : 10·8	6
16	...	0 41·7	44 56	21·80	268·9	6·56	10·5 : 11·5	5
17	...	0 42·2	45 45	21·80	266·7	7·10	10·5 : 12·0	3
18	...	0 42·6	45 16	21·80	103·0	12·32	11·0 : 12·0	3
19	...	0 44·8	44 56	21·80	330·7	4·94	10·5 : 11·5	5
20	439	0 46·1	27 19	26·91	297·6	4·45	6·0 : 6·0	10
21	53° 180	0 52·1	53 39	21·88	196·4	5·82	9·0 : 12·0	2
22	...	0 57·4	53 17	21·88	67·9	7·08	12·0 : 12·1	3
23	...	0 58·1	60 41	21·84	321·0	8·49	12·0 : 12·5	3
24	59° 171	0 58·9	60 9	21·84	118·4	7·75	9·8 : 10·0	3
25	...	0 59·4	60 29	21·84	260·5	13·17	10·0 : 10·2	3
26	...	1 0·8	60 42	21·84	148·0	10·76	10·0 : 11·5	3
27	...	1 3·0	60 34	21·84	89·8	13·73	10·0 : 12·0	3
28	59° 184	1 3·0	60 16	21·84	175·5	9·54	9·5 : 11·5	3
29	...	1 3·8	60 41	21·84	15·2	10·55	11·0 : 11·5	3
30	59° 188	1 3·9	59 46	21·84	315·6	8·58	8·0 : 13·0	3
31	...	1 4·7	59 50	21·84	129·7	6·26	11·0 : 11·0	3
32	59° 197	1 6·5	60 5	21·84	275·3	10·69	10·0 : 12·0	3
33	...	1 6·9	60 59	21·84	131·9	12·25	11·0 : 11·5	3
34	60° 179	1 7·5	60 49	21·84	174·3	5·54	9·0 : 12·0	3
35	29° 250	1 25·7	30 3	22·96	358·8	5·24	9·8 : 10·0	7
36	51° 334	1 30·4	51 48	25·88	90·4	10·42	9·0 : 11·0	5
37	842	1 34·7	45 38	23·92	227·7	14·78	9·0 : 12·0	10
38	...	1 36·1	45 20	22·80	286·8	3·32	12·0 : 12·5	6
39	...	1 38·1	44 45	22·80	257·6	4·17	11·5 : 11·7	5
40	878	1 39·0	52 56	23·83	241·4	18·21	9·0 : 10·0	8
41	53° 377	1 39·9	53 42	21·87	302·8	11·48	8·5 : 11·0	3

No.	Star (β.G.C. or B.D.)	R.A. (1930)	Dec. (1930)	Epoch 1900 +	Position Angle	Distance	Magnitudes	No. of Observations
		h m	° ′		°	″		
42	...	1 40·5	54 4	21·87	270·6	7·12	11·5 : 11·6	2
43	...	1 41·6	45 14	22·80	268·3	3·20	12·0 : 12·5	5
44	...	1 42·9	45 4	22·80	202·2	3·94	11·5 : 12·5	4
45	...	1 46·2	53 51	21·87	61·4	6·98	11·5 : 11·8	1
46	...	1 59·2	39 0	25·88	51·8	8·05	10·5 : 11·0	3
47	...	2 9·5	58 52	21·85	47·3	8·14	10·0 : 10·3	3
48	...	2 11·9	59 2	21·85	182·1	4·06	10·0 : 12·0	3
49	58° 429	2 12·5	59 0	21·85	138·9	11·66	9·7 : 10·0	3
50	...	2 13·9	59 10	21·85	45·0	7·95	10·0 : 10·5	3
51	1193	2 14·5	59 42	21·85	199·9	63·13	6·1 : 6·5	3
52	...	2 15·0	58 59	21·85	132·5	13·80	10·3 : 11·0	3
53	...	2 15·1	59 2	21·85	116·9	15·33	10·0 : 10·1	3
54	1209	2 15·8	−3 18	4·76	79·4	116·34	Var. : 10·0	7
				7·72	78·4	116·59	...	4
55	1239	2 19·5	15 6	23·01	260·2	34·63	8·7 : 9·0	5
56	...	2 25·7	57 40	21·98	341·9	8·35	11·5 : 11·6	3
57	1286	2 26·8	57 23	21·98	331·9	13·13 AB	7·5 : 11·5	3
				21·98	267·2	25·14 AC	7·5 : 11·8	3
58	1297	2 28·5	57 13	21·98	19·1	23·26	8·5 : 9·0	3
59	1310	2 29·4	31 6	22·88	41·9	23·21	9·0 : 12·0	6
60	...	2 30·5	56 45	21·98	173·3	4·48	11·2 : 11·5	3
61	1466	2 48·5	48 17	25·88	316·0	6·66	6·5 : 10·5	4
62	38° 626	3 0·0	38 28	5·06	26·3	12·78	9·3 : 10·5	7
				22·30	27·1	12·45	...	8
63	59° 593	3 1·7	60 13	22·45	32·5	6·12	9·5 : 12·0	5
64	...	3 3·1	60 13	22·45	143·0	2·99	10·5 : 11·0	6
65	38° 645	3 3·9	38 54	5·06	24·6	10·51 AB	9·0 : 10·5	7
				22·60	24·5	10·28 AB	...	6
				5·06	224·6	13·48 AC	9·0 : 11·0	4
				22·60	223·7	13·32 AC	...	5
66	60° 639	3 6·5	60 41	22·45	80·8	5·05	9·4 : 11·5	5
67	59° 614	3 10·3	59 57	21·85	290·5	9·26	9·0 : 12·0	1
68	1610	3 10·7	59 47	22·45	143·7	6·84	7·7 : 8·0	2
69	60° 661	3 11·9	60 53	22·45	253·1	13·78	9·8 : 12·0	4
70	1733	3 27·1	27 20	26·93	271·7	11·20	6·5 : 7·0	7
71	1854	3 42·1	41 17	4·45	47·3	8·35	8·2 : 8·8	8
				25·08	47·7	8·21	...	6
72	33° 815	4 9·1	33 14	21·91	325·6	11·46	9·7 : 10·0	4
73	32° 754	4 10·6	32 52	21·91	158·4	7·91	9·2 : 11·5	3
74	2210	4 24·5	30 13	25·11	35·2	14·45	6·0 : 8·0	9
75	44° 1021	4 42·7	45 4	23·75	21·5	6·44	9·2 : 12·0	6
76	44° 1029	4 44·0	44 41	23·75	84·3	10·80	9·8 : 12·5	3
77	45° 990	4 45·7	46 1	4·46	83·5	13·13	9·8 : 10·0	8
				27·02	83·1	12·83	...	2
78	...	4 53·8	50 3	21·99	154·5	8·74	11·2 : 11·5	2
79	2446	4 56·0	50 9	21·99	128·5	5·01	8·7 : 9·0	3
80	...	4 59·8	49 56	21·99	235·4	13·55	9·5 : 10·0	3

No.	Star (β.G.C. or B.D.)	R.A. (1930)	Dec. (1930)	Epoch 1900 +	Position Angle	Distance	Magnitudes	No. of Observations
		h m	° ′		°	″		
81	...	5 1·3	51 15	21·99	132·8	5·10	10·2 : 10·5	3
82	2504	5 2·0	50 12	21·99	303·4	17·86 AB	7·5 : 10·5	3
				21·99	63·1	25·07 AC	7·5 : 11·0	2
83	...	5 15·0	45 28	24·08	43·4	11·32	10·0 : 10·5	4
84	2758	5 25·6	29 30	23·97	350·2	14·80	7·0 : 8·9	6
85	53° 929	5 33·8	53 50	21·99	39·1	9·91	9·4 : 9·5	3
86	44° 1265	5 37·1	45 0	24·04	235·6	13·36	10·0 : 11·5	3
87	44° 1284	5 42·9	44 20	24·04	102·8	4·62	9·8 : 10·0	4
88	3070	5 53·9	0 24	23·10	194·3	57·56	8·0 : 8·2	6
89	3176	6 5·0	24 26	3·15	187·8	30·96	7·0 : 8·3	3
				24·08	188·1	30·94	...	3
90	3198	6 6·8	24 54	3·15	354·2	5·89	8·3 : 9·9	6
				24·08	355·6	6·23	...	6
91	3377	6 23·7	30 28	25·09	301·8	11·62	8·7 : 10·0	5
92	3399	6 25·8	30 29	25·09	353·1	4·17	8·8 : 9·1	5
93	3433	6 28·5	29 48	23·10	8·6	11·60	10·5 : 11·5	9
94	3797	7 0·0	20 40	4·81	83·7	86·52 } AB	3·7 : 10·5	7
				24·23	83·7	86·42 }	...	5
				24·23	349·3	95·62 AC	3·7 : 8·0	5
95	...	7 0·3	21 29	4·81	106·3	4·47	11·0 : 11·5	2
				24·23	106·3	4·69	...	2
96	3810	7 1·1	20 7	4·81	155·6	4·69	9·3 : 10·0	6
				24·23	157·0	4·89	...	3
97	15° 1531	7 14·4	15 31	25·17	135·6	15·28	9·5 : 11·5	6
98	...	8 56·5	41 52	6·25	326·5	3·90	10·0 : 11·5	11
99	5429	10 23·1	30 2	24·15	121·4	28·72	8·0 : 10·0	6
100	30° 2019	10 24·8	30 6	23·20	212·0	6·89	10·3 : 10·5	9
101	5533	10 39·1	44 59	24·27	9·4	1·79	8·5 : 8·8	2
102	5545	10 41·2	45 20	24·27	293·8	3·41	8·0 : 10·5	2
103	5779	11 23·3	3 24	5·77	149·8	28·98	6·3 : 7·3	11
				25·28	149·3	28·55	...	8
104	5790	11 24·3	3 15	5·77	174·0	91·85	5·0 : 7·0	8
				25·28	175·2	90·59	...	8
105	5803	11 26·4	3 36	5·77	6·0	3·55	8·0 : 10·2	8
106	5887	11 40·1	44 34	24·28	203·5	6·59 AB	9·8 : 10·0	5
				24·28	274·4	43·93 AC	9·8 : 10·2	5
107	6075	12 9·0	11 10	5·86	7·2	5·59	9·2 : 9·7	8
108	6088	12 11·5	10 22	5·86	64·7	26·02	8·5 : 8·5	8
109	29° 2368	13 3·5	28 59	25·30	155·7	18·55	9·5 : 9·6	6
110	29° 2373	13 5·9	29 3	25·30	226·8	23·08	8·5 : 10·5	6
111	6482	13 21·1	55 17	3·31	148·0	14·20	2·1 : 4·2	2
112	45° 2134	13 48·0	44 42	25·36	255·5	3·08	9·0 : 10·0	5
113	7043	14 48·9	44 53	25·32	125·9	6·08	8·0 : 10·5	6
114	14° 2850	15 10·5	14 42	25·33	11·4	4·80	8·5 : 10·0	4
115	...	15 13·0	−1 32	8·20	120·6	2·36	10·0 : 10·5	3
116	7332	15 33·6	40 2	4·28	327·3	121·74	7·0 : 7·5	8

No.	Star (β.G.C. or B.D.)	R.A. (1930)	Dec. (1930)	Epoch 1900 +	Position Angle	Distance	Magnitudes	No. of Observations
		h m	° ′		°	″		
117	31° 2891	16 39·9	31 21	4·21	226·9	11·46	9·3 : 11·0	8
118	8162	17 43·7	27 46	4·46	243·9	32·61	3·5 : 9·5	8
				28·51	244·3	33·40	...	8
119	45° 2600	17 46·3	45 7	25·77	76·2	17·54	9·5 : 9·6	3
120	8257	17 54·8	29 30	25·70	188·9	11·55	8·5 : 9·0	6
121	8279	17 56·3	30 3	25·70	278·0	19·50	7·0 : 8·0	6
122	10° 3696	18 48·0	10 17	7·34	245·4	13·61	9·7 : 11·0	3
123	8870	18 48·2	10 14	7·34	203·8	22·22	9·0 : 9·3	8
124	8871	18 48·3	10 14	7·34	220·0	11·88	9·0 : 10·5	3
125	5° 3994	18 54·6	5 54	5·05	104·8	5·54	9·4 : 10·5	8
126	9139	19 11·4	15 15	25·68	123·3	9·90	8·5 : 11·5	5
127	9149	19 12·1	14 58	25·68	267·1	89·40	5·5 : 7·5	9
128	9160	19 13·1	15 26	25·68	230·3	9·81	9·4 : 10·5	6
129	...	19 30·3	30 21	22·72	36·4	15·79	10·0 : 11·5	4
130	...	19 33·0	0 6	21·73	47·5	5·92	10·0 : 12·0	1
131	...	19 34·0	30 44	5·13	344·2	4·75	9·5 : 10·5	8
				22·72	342·7	4·85	...	4
132	+0° 4264	19 34·4	0 37	21·73	101·4	10·93	9·5 : 11·0	1
133	9481	19 35·0	30 11	5·13	351·9	6·50	8·9 : 8·9	8
				22·72	352·4	6·43	...	4
134	...	19 45·2	45 5	21·70	192·6	7·59	10·0 : 12·0	2
135	...	19 45·4	53 47	22·70	191·2	9·51	10·0 : 11·0	5
136	9675	19 47·9	44 58	23·68	326·2	30·87 AB	8·2 : 9·2	5
				23·68	137·6	8·28 AC	8·2 : 11·2	3
137	...	19 48·1	44 36	21·70	332·3	10·21	10·0 : 12·5	2
138	...	19 48·2	45 30	21·70	118·9	3·83	11·5 : 11·6	3
139	9694	19 48·5	54 28	22·70	148·9	11·64	8·0 : 10·4	5
140	...	19 48·7	44 34	21·70	46·6	9·98	12·0 : 12·5	3
141	...	19 49·9	45 38	21·70	317·1	12·13	11·0 : 11·5	3
142	53° 2311	19 49·9	53 28	22·70	319·7	10·14	9·2 : 9·3	5
143	...	19 50·0	45 7	21·70	253·1	14·49	10·0 : 13·0	2
144	...	19 51·6	44 25	23·68	117·9	20·94	9·5 : 9·6	5
145	...	20 9·6	15 35	21·77	329·5	9·09	10·0 : 12·0	2
146	...	20 12·3	14 49	21·77	296·9	5·52	12·0 : 12·2	2
147	...	20 13·7	14 52	21·77	213·6	8·97	11·0 : 12·0	2
148	...	20 14·5	15 25	21·77	107·3	8·40	10·0 : 11·5	2
149	...	20 14·9	15 40	21·77	20·9	4·10	10·0 : 10·5	2
150	...	20 14·9	15 5	21·77	149·6	8·88	10·0 : 11·5	2
151	37° 3949	20 26·2	37 56	23·68	121·8	4·06	9·2 : 10·0	8
152	10269	20 26·6	37 17	23·68	215·3	4·13 AB	8·4 : 8·8	7
				23·68	201·1	11·95 CD	8·9 : 10·4	8
153	10279	20 27·1	37 53	23·68	31·9	17·14	8·0 : 8·2	8
154	38° 4119	20 27·4	38 18	23·68	269·0	11·62	9·0 : 11·0	8
155	38° 4133	20 29·1	38 17	23·68	356·0	10·07 AB	9·8 : 11·0	5
				23·68	7·1	6·57 AC	9·8 : 11·5	5
156	...	20 29·6	37 12	23·68	87·8	5·84	10·0 : 11·0	7
157	10527	20 44·4	34 7	7·83	121·7	9·81	7·0 : 12·0	6
				25·21	121·2	9·83	...	2

No.	Star (β.G.C. or B.D.)	R.A. (1930)	Dec. (1930)	Epoch 1900 +	Position Angle	Distance	Magnitudes	No. of Observations
		h m	° ′		°	″		
158	10543	20 45·0	45 19	21·70	76·8	13·69	7·0 : 12·2	1
159	10558	20 46·6	45 51	21·70	172·4	20·87	6·0 : 11·0	4
160	...	20 46·7	45 9	21·70	313·7	3·59	11·0 : 12·0	2
161	44° 3596	20 47·6	45 5	21·70	344·8	8·53	9·0 : 12·0	3
162	44° 3598	20 47·9	44 49	21·70	31·5	12·18	9·2 : 10·5	4
163	10585	20 48·9	45 2	21·70	249·6	9·41	9·1 : 9·3	7
164	...	20 49·5	45 21	21·70	260·2	7·56	10·5 : 12·5	2
165	...	21 0·5	30 9	22·78	145·1	7·01	11·5 : 12·0	5
166	10706	21 0·8	30 47	22·78	151·7	4·83	7·2 : 11·3	6
167	10732	21 3·6	38 23	5·87	126·6	22·58	5·3 : 5·9	4
				30·63	134·4	25·19	...	4
168	...	21 7·5	15 23	21·74	6·4	9·26	11·5 : 12·0	1
169	15° 4361	21 7·6	15 59	21·74	332·7	11·34	9·2 : 11·5	2
170	...	21 8·0	15 29	21·74	45·7	8·26	12·0 : 12·1	2
171	...	21 8·1	15 29	21·74	262·1	6·57	12·0 : 12·1	2
172	...	21 8·9	14 54	21·74	276·5	11·74	10·0 : 10·5	2
173	...	21 9·0	38 0	23·82	142·4	4·01	10·0 : 11·0	4
174	...	21 9·5	38 11	23·82	117·7	10·48	10·0 : 11·0	5
175	...	21 9·9	14 43	21·74	36·5	7·88 AB	10·0 : 10·5	2
				21·74	151·5	11·36 AC	10·0 : 11·0	2
176	...	21 10·0	14 43	21·74	254·5	3·69	10·0 : 12·5	2
177	...	21 10·2	15 47	21·74	8·3	8·70	12·0 : 12·5	1
178	...	21 10·5	37 26	23·82	175·4	12·25	9·5 : 11·0	3
179	36° 4469	21 10·5	37 16	6·90	140·8	3·41	9·0 : 10·5	5
				23·82	142·7	3·99	...	7
180	...	21 10·5	37 27	23·82	153·6	7·24	10·5 : 11·5	3
181	...	21 10·6	38 18	23·82	126·2	17·14	9·5 : 11·5	3
182	...	21 11·0	37 55	23·82	320·4	16·56	11·0 : 11·5	3
183	10829	21 11·1	9 43	4·53	17·6	44·48	4·1 : 10·2	3
184	...	21 11·2	37 37	23·82	204·7	10·56	10·0 : 10·1	7
185	15° 4378	21 11·5	15 42	21·74	84·4	4·71	9·5 : 12·0	2
186	...	21 11·6	37 46	23·82	133·5	10·56	10·0 : 10·5	6
187	...	21 12·6	37 27	23·82	5·1	4·41	11·0 : 11·5	3
188	...	21 13·8	38 0	23·82	327·9	12·52	11·0 : 11·5	2
189	...	21 13·9	38 8	23·82	356·6	3·53	10·5 : 11·5	1
190	37° 4263	21 15·5	37 31	23·82	268·1	14·87	9·2 : 10·2	6
191	...	21 20·4	60 19	21·79	46·7	7·24	11·0 : 12·0	1
192	59° 2375	21 23·8	59 39	22·30	115·1	7·47	9·4 : 11·0	5
193	...	21 24·0	59 51	22·30	71·6	5·24	9·2 : 10·5	4
194	...	21 24·9	60 42	21·79	256·7	4·52	10·5 : 12·0	3
195	59° 2394	21 28·8	59 56	22·30	56·3	6·21	10·0 : 10·2	5
196	...	21 28·9	60 3	22·30	64·6	11·57	10·5 : 10·5	5
197	11033	21 29·0	41 55	22·81	319·8	6·49	7·5 : 11·0	3
198	...	21 36·8	0 15	21·77	125·0	4·17	10·0 : 11·0	3
199	...	21 38·8	0 18	21·77	313·8	15·15 AB	10·0 : 10·2	3
				21·77	98·8	19·54 AC	10·0 : 10·5	3

No.	Star (β.G.C. or B.D.)	R.A. (1930)	Dec. (1930)	Epoch 1900 +	Position Angle	Distance	Magnitudes	No. of Observations
		h m	° ′		°	″		
200	11175	21 38·9	0 7	21·77	154·6	26·01	8·2 : 8·5	3
201	30° 4597	22 0·8	30 32	21·67	339·0	5·34	8·2 : 12·0	2
202	...	22 2·6	30 39	21·67	275·9	5·87	10·0 : 12·5	2
203	...	22 14·7	15 57	22·82	14·5	3·36	10·0 : 12·5	2
204	11645	22 16·5	15 30	22·82	246·4	22·04	9·0 : 12·0	2
205	...	22 23·9	56 52	22·70	308·7	9·69	10·0 : 10·5	3
206	11753	22 25·1	57 8	22·70	208·4	14·27	10·5 : 11·5	1
207	11778	22 27·7	56 52	22·70	356·5	14·68	6·7 : 10·0	3
208	11779	22 27·8	53 10	26·91	69·6	4·66	8·0 : 8·0	9
209	42° 4424	22 28·0	42 33	22·85	358·3	9·46 AB	9·4 : 10·2	3
				22·85	340·7	9·12 AC	9·4 : 10·2	3
210	...	22 29·0	56 49	22·70	206·0	7·01	10·5 : 11·0	2
211	...	22 29·0	42 6	22·85	302·2	6·65	11·0 : 11·5	2
212	...	22 29·5	42 47	22·85	287·5	7·00	9·0 : 11·5	4
213	42° 4445	22 31·0	43 19	22·85	327·2	5·84	8·7 : 9·2	4
214	11894	22 37·6	0 31	22·81	136·9	3·45 AB	11·0 : 11·5	3
				22·81	78·6	22·35 AC	11·0 : 11·5	2
215	...	22 46·4	45 5	21·73	31·5	10·20	9·5 : 9·8	4
216	...	22 48·2	45 37	21·73	255·2	9·63	10·5 : 12·0	4
217	12014	22 48·6	29 39	21·83	181·9	11·29	9·0 : 9·5	3
218	...	22 49·4	45 21	21·73	310·8	14·33 AB	10·0 : 12·0	4
				21·73	283·9	14·40 AC	10·0 : 12·5	4
219	12031	22 50·2	30 24	21·83	279·3	24·38	7·4 : 10·5	3
220	...	22 52·1	29 32	21·83	62·6	3·73	11·0 : 11·5	2
221	...	22 52·7	44 46	21·73	279·5	13·01	9·5 : 10·0	4
222	12227	23 9·7	55 19	21·95	323·0	7·14	9·5 : 10·0	3
223	...	23 10·6	54 35	21·95	275·5	7·91	10·5 : 12·0	2
224	12258	23 12·2	54 37	21·95	260·7	16·65 AB	8·5 : 11·5	3
				21·95	112·3	5·52 BC	11·5 : 12·0	2
225	...	23 12·3	14 51	21·77	272·4	10·76	10·0 : 11·5	3
226	15° 4791	23 12·5	15 29	21·77	177·5	7·96	9·5 : 12·5	2
227	...	23 15·0	14 33	21·77	7·2	9·21	10·0 : 11·0	3
228	...	23 15·7	55 27	21·95	327·2	9·74	11·5 : 12·0	2
229	...	23 15·7	44 5	23·81	280·8	8·51	11·0 : 11·5	1
230	...	23 15·8	54 23	21·95	70·0	11·27	9·0 : 12·0	2
231	42° 4636	23 16·0	43 20	4·90	32·4	6·50	9·5 : 10·5	5
				23·55	32·2	6·52	...	8
232	12313	23 17·4	43 45	4·74	178·1	13·50	6·5 : 10·0	5
				23·55	176·6	13·55	...	4
233	...	23 19·6	60 27	21·74	252·5	8·88	10·0 : 11·5	3
234	42° 4655	23 19·7	43 22	23·81	148·4	3·15	9·5 : 11·0	4
235	...	23 19·8	60 24	21·74	53·0	11·43	10·0 : 10·2	5
236	...	23 20·1	60 10	21·74	208·8	7·05	10·0 : 12·5	2
237	...	23 20·5	60 22	21·74	83·2	9·21	10·5 : 12·5	3
238	59° 2722	23 22·8	59 51	21·74	105·3	4·78	10·0 : 11·5	3
239	...	23 22·8	60 40	21·74	81·8	7·17	12·5 : 12·5	3
240	...	23 23·2	59 35	21·74	198·0	6·63	10·0 : 12·0	3
241	...	23 23·2	60 41	21·74	90·7	10·57	12·0 : 12·2	3

No.	Star (β.G.C. or B.D.)	R.A. (1930)	Dec. (1930)	Epoch 1900 +	Position Angle	Distance	Magnitudes	No. of Observations
		h m	° ′		°	″		
242	...	23 24·1	59 40	21·74	77·8	10·13	11·0 : 11·5	3
243	59° 2727	23 24·2	60 3	21·74	136·5	11·83	9·3 : 12·5	3
244	...	23 24·3	60 4	21·74	342·7	5·91	11·0 : 11·5	3
245	...	23 24·9	59 57	21·74	35·3	9·14 AB	10·0 : 11·0	4
				21·74	99·2	10·71 AC	10·0 : 12·0	4
246	59° 2732	23 25·8	59 34	21·74	336·7	6·24	10·0 : 13·0	3
247	59° 2733	23 26·3	59 45	21·74	323·3	3·29	10·0 : 10·5	5
248	...	23 28·2	60 53	21·74	251·3	9·84	11·0 : 11·5	3
249	...	23 32·2	54 44	21·96	255·4	5·36	10·5 : 12·0	3
250	...	23 33·6	54 52	21·96	110·5	8·61	10·0 : 10·3	3
251	...	23 35·8	54 53	21·96	84·7	6·66	10·5 : 11·5	3
252	53° 3211	23 36·0	54 21	21·96	167·1	7·14	9·4 : 11·0	3
253	12503	23 36·6	54 48	21·96	253·3	15·87	8·8 : 10·0	3
254	...	23 37·4	55 16	21·96	302·6	3·92	11·0 : 12·0	2
255	...	23 37·6	54 41	21·96	121·8	4·13	10·0 : 11·5	2
256	54° 3028	23 38·5	54 25	21·96	273·3	3·25	9·4 : 11·0	2
257	...	23 49·5	44 27	21·74	294·7	12·38	11·0 : 12·5	3
258	12601	23 49·6	37 39	26·93	89·1	5·27	7·0 : 7·0	7
259	44° 4516	23 51·3	45 23	21·74	78·6	13·59	8·8 : 12·0	3
260	...	23 51·5	44 24	21·74	312·1	5·17	11·5 : 12·5	3
261	45° 4359	23 51·6	45 38	21·74	21·1	8·42	9·6 : 10·0	3
262	...	23 52·0	44 45	21·74	18·5	5·78	9·5 : 12·5	3
263	...	23 53·7	45 25	21·74	219·1	6·05	11·0 : 12·5	3

MICROMETRICAL MEASURES OF DOUBLE STARS.

(23RD SERIES.)

T. E. Espin and W. Milburn.

The micrometrical measures are given as in previous lists, those marked E being made with the 24-inch, and the ones marked M with the $17\frac{1}{4}$-inch.

Name	1900 R.A.	1900 Decl.	P.A.	Dist.	Mags.	Nights	Date 1900 +	
	h m	° ′	°	″				
Σ 42	0 30·7	+29 27	28·1	5·90	...	2	30·913	M
Espin 316	47·3	32 43	292·4	2·20	9·3 11·3	3	30·033	E
Barton 3	49·9	26 37	341·4	2·87	10·0 10·5	3	31·888	M
h 629	50·2	34 1	69·6	9·91	9·5 11·0	1	30·055	E
Espin 317	54·9	31 56	195·1	6·66	9·3 9·9	1	30·038	E
A 1513	55·9	39 4	292·5	3·75	9·1 10·5	2	31·394	E
Scheiner 12*	59·9	31 32	62·1	4·49	9·4 9·5	3	30·933	E
Barton 232	1 15·3	31 55	309·5	3·41	9·7 10·5	1	31·014	E
Barton 6	42·0	27 41	37·2	2·72	10·5 10·6	2	31·010	M
h 1134	3 12·1	28 0	50·6	11·64	9·8 10·7	2	30·490	M
OΣ 245	12 12·5	29 29	280·0	8·12	...	3	31·345	M
h 3339	24·7	29 10	158·2	11·68	9·5 10·5	2	31·339	M
Σ 1700	53·9	27 40	85·0	6·70	...	2	31·349	M
Espin 440	13 12·6	29 12	178·4	2·45	9·5 10·0	3	31·345	M
Σ 1810	14 7·1	28 30	177·0	2·61	...	3	31·345	M
Σ 1816	9·5	29 34	90·4	1·88	...	2	31·349	M

Name	1900 R.A. h	m	1900 Decl. °	′	P.A. °	Dist. ″	Mags.		Nights	Date 1900 +		
h 544	14	13·7	28	50	325·6	8·39	...		2	31·339	M	
ϱ Herculis	17	20·2	37	14	315·9	4·04	5·0	5·1	2	31·645	E	
Espin 2169		46·1	36	5	336·9	8·03	9·5	9·6	3	31·599	E	
Σ 2263		56·9	26	33	162·2	7·66	8·5	9·2	2	31·590	M	
Espin 470		59·3	27	51	209·1	7·89	...		2	31·590	M	
Espin 472	18	9·9	27	57	15·4	3·25	9·5	10·0	2	31·590	M	
h 1324		22·6	28	38	16·4	5·40	9·5	9·8	2	30·617	M	
Espin 476*		26·7	28	0	349·7	2·56	...		2	30·617	M	AB
Espin 241		48·0	36	41	75·9	2·88	9·5	10·5	3	31·679	E	
Espin 349	19	6·8	31	35	218·9	6·16	9·4	12·0	1	31·616	E	
Espin 2113		15·7	37	4	352·1	5·01	10·0	10·0	2	31·707	E	
h 1395		21·2	36	54	63·8	3·00	9·0	9·0	1	31·684	E	
Espin 354		44·5	31	29	325·4	8·22	9·0	12·0	2	31·652	E	
Milburn 657		48·0	29	10	224·9	3·14	11·5	11·5	2	31·638	M	
Espin 356		48·3	31	24	343·9	5·95	9·3	9·5	1	31·813	E	
J 782		53·1	31	53	61·1	2·25	9·5	9·5	1	31·857	E	
Espin 358		59·7	31	33	198·6	7·37	9·0	11·2	1	30·840	E	
h 1473	20	1·3	26	59	143·5	10·93	9·5	9·6	3	31·783	M	
Scheiner 998		8·5	37	52	106·3	3·34	10·5	11·0	3	31·830	E	
Scheiner 1050		12·5	37	47	312·7	4·42	10·6	10·7	2	31·691	E	
Espin 206		25·1	37	50	118·9	3·98	9·1	9·4	4	31·784	E	
Weisse 35		25·4	37	11	224·2	3·58	8·7	8·9	1	31·801	E	AB
					201·4	11·24	9·0	10·5	1	31·801	E	CD
Barton 50		30·8	27	33	200·6	4·13	9·5	9·6	2	31·708	M	
h 1544		31·5	27	33	52·7	8·04	9·5	9·6	3	31·698	M	
Scheiner 1238		41·5	36	29	299·1	7·60	10·0	10·1	2	31·824	E	
A.G. 265		41·8	36	25	25·3	6·48	9·0	9·1	3	31·838	E	
Espin 369		44·6	31	37	306·9	6·40	9·3	11·5	3	30·771	E	
Scheiner 1300		49·9	36	39	301·3	2·10	9·4	9·6	3	31·833	E	
Scheiner 1327		54·1	36	51	81·4	1·70	9·6	10·2	2	31·908	E	
Milburn 486		56·4	28	20	350·7	4·02	...		2	31·683	M	
h 1611	21	2·7	27	52	296·7	9·50	9·5	9·8	3	31·845	M	
Scheiner 1506		16·0	39	32	80·6	5·13	9·5	9·6	3	31·712	E	
Ku 61		23·8	38	8	271·5	3·54	9·5	9·6	1	31·912	E	
Espin 379		28·9	29	48	303·2	8·77	9·0	10·5	2	31·735	M	
Espin 380*		30·4	29	50	315·9	2·37	10·5	10·7	2	31·735	M	
Espin 2321		47·8	32	24	94·0	2·37	9·3	10·5	2	30·868	E	
Ho 174		51·5	36	46	87·3	5·75	...		1	31·715	E	CD
Espin 384		56·4	31	41	65·1	3·12	9·5	9·6	1	30·857	E	
Ku 63	22	8·1	33	28	242·4	4·29	9·3	9·4	3	30·862	E	
Milburn 582		27·2	29	22	341·2	4·60	9·5	10·0	1	30·925	M	
h 968		36·0	36	22	107·2	4·52	9·2	10·0	2	31·791	E	
β 176		38·0	38	46	47·5	2·16	9·0	9·1	3	31·823	E	
Espin 1997		39·1	37	41	92·6	4·24	10·5	10·7	2	31·950	E	
Σ 2942		39·6	38	56	280·1	2·29	6·5	9·2	3	31·860	E	
Espin 1999		59·5	37	20	240·6	3·37	10·0	10·0	1	31·783	E	
Σ 3042	23	46·8	37	21	87·0	4·41	7·5	7·7	1	31·958	E	
Σ 3050		54·4	33	10	238·3	1·58	6·0	6·0	1	30·838	E	

Notes on stars marked *

Scheiner 12. Angle slowly increasing.

Espin 476. *Comes*, 10·5 mag., P. 324°·5, D. 28″ ± 1930·619 M.

Espin 380. AB, P. 62°·4, D. 15″ ±, 9·1 ; AC, P. 108°·8, D. 56″ ±, M.

New Double Stars

The observation of the B.D. stars between +65° and +30° being completed in certain parts of the sky, it seemed well to continue the work of Scheiner, *Katalog von Doppelsternen der Photographischen Himmelskarte aus der zone von* +31° *bis* +40° *Deklination* (Publikationen zu Potsdam, Nr. 59). This was made possible by the receipt of the charts made at Uccle under the direction of Professor Stroobant, which he has kindly sent to us. The following list contains many of these objects found on the charts, and measured with the 24-inch reflector.

Espin No.	B.D.	1900 R.A.	1900 Decl.	P.	D.	Mags.		Nights	Date 1931	
	°	h m	° ′	°	″					
2445	...	0 8·3	+37 28	78·1	8·28	10·0	10·2	3	·801	
2446	...	8·9	34 23	90·4	6·43	10·0	11·5	2	·447	
2447	+37·80	24·9	37 44	246·4	10·59	9·3	10·0	3	·886	
2448	...	57·6	39 8	57·9	3·81	10·0	11·0	2	·824	
2449	...	1 21·3	37 11	178·0	6·81	10·5	10·7	3	·912	
2450	...	27·9	37 50	258·7	10·41	10·0	10·5	2	·888	
2451	...	2 10·1	37 22	55·3	4·88	10·7	11·2	2	·935	
2452	...	21·0	37 50	200·8	7·01	11·0	11·0	2	·935	
2453	...	28·9	37 55	73·8	7·78	11·0	11·5	2	·935	
2454	...	43·6	37 59	177·4	7·74	9·5	10·0	3	·937	
2455	...	47·1	37 41	202·8	7·49	9·8	11·0	2	·949	
2456*	+32·567	3 1·8	32 28	104·7	1·95	9·5	11·0	2	·514	
2457	...	5·9	37 45	26·1	3·93	10·0	12·0	2	·935	
2458	...	25·3	37 42	39·5	3·76	10·0	10·1	3	·936	
2459*	+35·759	46·4	35 14	213·6	4·66	9·2	10·0	1	·958	
2460*	+35·765	47·3	35 22	123·8	2·96	9·5	9·5	1	·958	
2461	...	57·8	37 30	45·6	7·89	10·0	10·2	2	·925	
2462*	...	4 6·3	37 36	176·9	4·12	11·0	11·5	2	·949	
2463	+35·862	17·3	35 21	153·6	4·49	9·5	12·0	2	·935	
2464	+34·898	32·7	34 15	289·4	7·37	9·5	13·5	1	·0684	
2465	+36·953	46·1	36 16	74·9	5·24	9·5	12·0	1	·9583	
2466	+34·1000	5 13·1	34 15	305·2	6·87	9·3	12·0	1	·068	
2467	...	6 22·6	34 39	267·4	4·66	10·5	10·5	1	·068	
2468	+34·1452	40·5	34 57	103·0	5·91	9·3	12·0	1	·068	
2469	+34·1537	7 2·6	34 23	310 ±	7 ±	9·5	13·0	1	·068	AB
				220 ±	9 ±		11·0	1	·068	AC
2470	...	12 2·2	33 26	136·0	4·16	9·5	11·0	1	·356	
2471	+33·2300	13 1·8	33 12	32·1	8·37	9·2	14·0	3	·370	
2472	+33·2330	16·7	33 15	86·7	7·01	9·2	13·5	2	·349	
2473	...	36·4	32 41	132·7	6·25	11·0	11·0	2	·378	
2474	+34·2470	50·9	33 57	339·2	2·62	9·7	10·7	2	·339	
2475	+32·2533	14 53·7	31 54	220·2	2·16	10·0	10·0	1	·381	
2476	...	15 9·4	33 6	215·7	2·79	10·5	11·5	1	·381	
2477	...	17 52·6	36 37	0·4	3·86	11·5	12·0	2	·590	
2478	...	56·1	38 3	245·7	5·21	10·5	11·0	2	·645	
2479	+36·3000	59·0	36 43	235·3	8·74	9·5	11·5	2	·590	
2480	...	18 6·7	37 12	351·4	5·08	10·5	11·0	3	·687	
2481*	...	30·0	37 22	155·1	3·46	10·5	11·0	4	·663	
2482	...	41·1	37 27	155·8	6·60	10·0	10·5	3	·646	

Espin No.	B.D.	1900 R.A.	1900 Decl.	P.	D.	Mags.	Nights	Date 1931	
	°	h m	° ′	°	″				
2483	...	18 41·4	36 45	277·8	5·04	10·5 11·0	2	·699	
2484	...	42·9	37 39	328·5	5·30	11·0 11·2	3	·704	
2485	...	51·6	36 37	164·4	5·99	11·0 11·5	3	·779	
2486	...	52·4	36 59	280·9	6·58	10·5 11·6	2	·782	
2487	...	19 12·1	37 37	266·0	4·83	10·0 10·2	2	·717	
2488	...	13·7	37 39	255·8	4·03	11·0 11·7	2	·717	
2489*	U Lyræ	16·6	37 42	181·6	9·37	Var. 13·0	3	·783	
2490	+37·3421	17·3	37 28	12·0	6·76	10·0 10·2	3	·804	BC
				40·3	18·72	A = 9·2	3	·804	AB
2491	...	17·7	37 9	357·1	4·91	10·7 10·8	3	·735	
2492	...	20·5	37 47	155·3	7·04	10·0 10·5	3	·748	
2493	...	23·8	37 15	307·9	7·01	9·7 10·8	3	·639	
2494	+36·3620	32·3	36 59	97·2	5·29	9·5 12·0	3	·793	
2495	...	33·9	37 57	147·3	5·89	10·5 11·0	3	·779	
2496	...	34·8	38 0	12·9	4·91	9·7 13·0	2	·694	
2497	...	35·2	37 28	247·3	5·51	9·5 11·0	2	·691	
2498	...	42·7	37 11	171·9	5·39	10·6 11·7	3	·769	
2499	...	45·4	37 16	148·2	6·06	10·0 11·5	2	·793	
2500	...	51·9	37 30	312·9	5·14	10·5 11·2	4	·772	
2501	...	59·4	37 25	124·6	2·96	10·5 11·0	4	·795	
2502	...	20 13·9	37 47	325·1	4·85	10·5 11·0	3	·694	
2503	...	14·7	36 40	94·6	3·43	10·5 10·7	4	·655	
2504	...	15·5	38 3	104·6	7·62	10·5 11·0	2	·835	
2505	+37·3889	16·0	37 24	253·0	6·49	9·0 12·5	3	·804	
2506	...	18·8	36 37	149·2	5·62	10·5 10·7	2	·694	
2507	...	24·3	37 53	84·8	7·72	10·5 11·0	2	·773	
2508	...	25·1	37 52	322·2	6·35	9·7 12·0	3	·743	
2509	...	25·9	37 13	242·6	3·13	10·0 10·5	5	·678	
2510	...	32·0	36 14	333·4	6·35	11·1 11·2	2	·876	
2511	...	32·4	37 51	257·5	5·63	11·0 11·2	5	·802	
2512	...	37·4	37 40	331·8	6·84	10·0 12·0	3	·676	
2513	...	41·1	36 29	280·5	6·76	11·0 11·5	3	·865	
2514	...	42·2	36 31	120·9	5·39	10·0 11·0	4	·873	
2515	+36·4228	42·4	36 24	168·6	6·60	9·3 12·5	3	·899	
2516	...	48·6	37 36	216·8	6·00	9·7 10·8	3	·722	
2517	...	21 18·1	37 8	113·9	4·62	10·5 11·0	3	·800	
2518	...	18·2	39 45	247·1	2·67	10·0 11·5	3	·728	
2519	...	19·3	37 35	93·0	5·78	10·5 10·8	2	·725	
2520	...	23·2	37 26	241·9	5·74	10·0 11·5	3	·781	
2521	+36·4660	38·8	37 6	196·7	7·51	9·0 12·0	2	·876	
2522	...	39·5	37 55	94·1	4·21	9·5 13·0	3	·909	
2523	...	51·5	36 40	28·4	3·91	11·0 11·0	2	·769	
2524	...	51·7	37 24	279·7	5·28	11·0 11·0	3	·763	
2525	...	52·2	36 56	106·6	7·26	9·5 11·0	2	·919	
2526	...	53·1	37 3	257·6	5·04	11·0 11·1	2	·927	BC
				87·3	16·66	A = 9·8	2	·927	AB
2527	...	54·6	37 16	323·8	7·73	10·5 10·7	3	·808	
2528	...	58·9	37 35	85·8	4·43	10·5 11·5	2	·769	
2529	...	59·3	37 56	270·6	4·60	11·0 11·0	2	·694	
2530	...	22 11·3	37 41	311·0	6·34	10·1 10·6	3	·670	
2531	...	16·7	37 14	238·4	3·13	11·0 11·0	2	·657	
2532	...	30·4	36 31	292·6	6·08	9·7 10·2	2	·739	

Espin No.	B.D.	1900 R.A.	1900 Decl.	P.	D.	Mags.		Nights	Date 1931
	°	h m	° ′	°	″				
2533	...	22 34·0	37 23	314·6	7·45	11·0	12·0	2	·950
2534	...	46·7	36 42	67·2	5·82	10·0	10·1	3	·650
2535	+37·4750	58·0	37 26	201·5	6·03	9·5	11·2	3	·774
2536	+37·4776	23 5·6	37 40	246·5	5·90	9·0	12·0	2	·950
2537*	...	11·6	37 7	54·7	5·58	11·0	11·2	1	·958
2538	...	12·4	37 23	38·1	4·39	10·0	12·0	4	·718
2539	...	14·6	37 53	47·0	6·00	11·0	11·5	3	·853
2540	...	16·6	36 55	314·1	6·04	10·0	10·0	2	·706
2541	...	46·6	38 11	271·3	6·72	10·5	11·0	2	·871
2542*	...	54·4	38 9	274·2	2·79	10·5	11·5	5	·847
2543*	...	54·9	39 31	251·6	4·83	11·0	12·0	5	·312

Notes on stars marked *

2456. M. 1931·994. P. 103°·8.
2459. M. 1931·994. P. 219°·6, D. 4″·21.
2460. M. 1931·994. P. 130°·9, D. 3″·12.
2462. M. 1931·994. P. 179°·8. Mags. 10·0, 10·2.
2477. SP β Lyræ.
2481. N. star of two.
2489. U Lyræ.—The star was noted on October 2 as 0·2 mag. above B.D. +37°·3430, mag. 8·4 in B.D., and was found to be a fine example of Type N (IV Type) in spectroscope.
2503. 3′ S. of 36 Cygni.
2506. 1′ N. of B.D. +36°·4035.
2509. 3′ N. of Weisse 35.
2513. S. star of a pair.
2517. 9·6 sec. P.B.D. +36° ·4531, 9·0 mag.
2521. A *comes* at P. 93°.
2537. M. 1931·994. P. 55°·9.
2542. N. star of pair.
2543. A 13 mag., P. 68°·5, D. 20″ ±.

Milburn No.	B.D.	1900 R.A.	1900 Decl.	P.	D.	Mags.		Nights	Date 1931
	°	h m	° ′	°	″				
674	+26·35	0 15·6	+27 3	265·4	4·90	9·5	9·7	1	·994
675	+26·39	16·1	27 11	97·1	7·81	8·5	10·5	1	·994
676	...	54·9	26 4	91·4	4·13	9·7	13·0	2	·953
677	...	59·5	27 28	216·7	8·08	9·8	10·2	2	·389
678	...	1 15·4	30 11	160·5	1·53	10·0	11·7	2	·847
679	...	36·2	29 52	240·0	5·32	9·6	9·8	2	·888
680	...	37·5	29 36	121·5	2·83	9·8	10·2	1	·994
681	...	2 4·1	29 57	87·3	5·54	9·6	10·7	4	·680
682	+28·373	8·5	28 22	293·1	4·36	9·5	11·0	2	·038
683	...	3 40·4	29 29	14·7	3·21	9·9	10·2	2	·925
684	+29·625	43·1	29 12	299·2	1·95	9·0	9·7	2	·949
685	...	4 7·0	27 11	214·9	2·37	10·0	10·6	3	·936
686	...	40·1	29 39	249·3	2·45	9·8	10·0	3	·655
687	+29·826	5 4·0	29 32	25·5	9·27	9·3	10·5	1	·958

Milburn No.	B.D.	1900 R.A.	1900 Decl.	P.	D.	Mags.		Nights	Date 1931	
	°	h m	° ′	°	″					
688	...	6 28·1	29 17	240·8	21·61	9·5	11·0	1	·068	AB
				195·8	6·90		11·2	1	·068	BC
689	...	13 8·0	28 14	162·4	2·22	11·0	11·5	2	·378	
690	+27·2518	15 32·4	27 46	232·4	2·64	9·5	9·6	2	·378	
691	...	18 29·3	28 30	17·0	3·56	9·9	12·2	3	·674	
692*	...	49·6	28 16	298·9	2·37	11·0	13·0	5	·663	
693	...	51·3	28 34	187·7	4·93	10·0	12·2	3	·686	AB
				155·1	7·70		11·5	2	·679	AC
694	...	19 9·3	27 9	111·8	3·40	10·0	10·2	4	·739	
695	+27·3299	9·9	27 43	50·8	8·29	9·5	11·5	3	·722	
696	...	25·6	28 54	65·7	3·14	11·0	14·0	5	·689	
697	...	26·3	28 58	3·3	3·14	10·7	12·3	3	·670	AB
				98·8	4·21		10·7	2	·670	AC
698	...	47·6	27 30	358·3	6·13	10·5	10·6	2	·735	
699	...	48·2	29 9	1·0	2·10	11·0	11·5	5	·678	
700	...	49·2	29 1	326·8	3·39	11·0	12·5	3	·667	
701	+26·3774	20 0·3	27 5	322·4	3·37	9·5	10·4	2	·761	
702	...	0·5	27 2	49·5	5·14	10·5	10·8	3	·778	
703	...	2·3	27 33	230·7	2·22	11·0	12·5	3	·791	
704*	...	2·6	28 39	117·3	2·68	11·2	13·0	3	·710	
705	...	4·9	27 30	171·0	2·87	9·6	13·0	2	·794	
706*	...	8·6	27 7	172·3	1·76	10·2	13·2	3	·815	
707*	...	20·5	27 20	2·4	2·76	10·0	11·5	2	·835	
708*	+28·3799	30·7	28 12	183·0	3·83	8·7	14·5	4	·728	
709	+27·3838	38·6	27 34	328·0	4·21	9·5	11·2	2	·786	
710	+28·3872	40·3	28 34	183·9	6·88	9·5	9·8	4	·655	
711	...	41·9	28 54	40·0	4·52	9·8	12·0	3	·646	
712	+27·3861	42·1	27 18	50·7	5·60	9·5	11·0	3	·888	AB
				131·9	21 ±		9·6	2	·876	AC
				88·4	5·17		11·0	2	·876	CD
713	...	44·3	27 11	352·9	6·74	9·6	10·0	2	·854	
714	...	50·2	27 13	206·8	3·71	9·8	10·9	3	·722	
715	+29·4321	21 3·5	29 22	36·2	3·60	8·5	12·7	3	·665	AB
				332·2	4·85		13·0	3	·665	AC
716	...	12·6	29 36	315·1	2·60	9·8	10·1	3	·772	
717	...	30·8	26 19	139·8	3·29	10·0	13·0	3	·927	
718	...	30·9	29 53	321·0	8·53	9·6	9·7	3	·772	
719	...	32·0	29 48	143·4	6·01	9·6	11·0	2	·776	
720	...	37·6	29 24	148·3	7·39	10·0	11·2	2	·876	
721	...	51·9	29 33	17·0	1·68	9·5	9·5	3	·694	AB
				206·1	5·25		12·0	3	·694	AC
722	...	22 1·9	27 37	273·8	4·48	10·0	10·1	3	·743	AB
				323·4	24 ±		10·0	1	·715	AC
723	...	7·4	25 16	344·9	2·68	10·0	10·5	1	·942	
724	...	21·3	30 13	236·4	2·95	10·5	11·5	4	·738	
725	...	34·1	29 29	8·6	6·28	9·5	9·8	2	·778	
726*	...	52·8	27 48	8·1	2·58	11·0	11·1	3	·698	
727*	...	54·9	25 21	4·4	3·29	10·5	11·0	2	·843	
728	...	23 32·4	25 31	32·5	2·40	10·7	12·5	3	·773	
729	...	32·8	25 14	173·2	5·67	9·8	11·0	2	·667	
730	...	35·2	26 8	314·8	4·50	10·0	10·1	3	·937	
731	...	41·8	25 34	242·2	3·94	9·6	11·0	2	·876	
732	...	56·6	26 14	326·1	2·98	10·0	11·5	2	·768	

*Notes on stars marked **

692. AC ; 300° ± ; 8″ ± ; C = 13·5.
704. 1928·816 ; 106°·5 ; 2″·83 ; 12·0, 14·0.
706. One of a group of faint stars = B.D. + 27°·3650.
707. At 249°·0 ; 110″ ± from B.D. + 27°·3720.
708. 1926·780 : close faint *comes* at 170° ±.
1926·838 : a 14·5 at 170° ± ; 3″ ±.
Examined on many other nights since 1926, but only verified in 1931.
726. 1926·923 ; 176°·0 : 2″–3″ ± : very faint pair.
727. At 303°·1 ; 64″ ± from B.D. + 25°·4854.

Missing Stars

	°	h m s	° ′			
B.D.	+29·490	2 43 33·4	+29 20·5	mag.	9·5.	1931 Nov. 30, Dec. 17. 13 mag. near place. M.
	+29·617	3 38 15·7	+29 48·4	,,	9·5.	1931 Dec. 10, 17. M.
	+36·590	4 42 47·9	+36 9·3	,,	9·3.	1931 Dec. 17. No 9·3 star here. E.
	+29·2549	14 27 12·3	+29 4·9	,,	9·5.	1931 May 6, 11. No star seen. M.
	+28·3102	18 45 12·0	+28 45·0	,,	9·4.	1931 Aug. 10, 18. Not seen. M.
	+29·4371	21 11 48·3	+29 12·4	,,	9·4.	1931 Oct. 7. Not seen. M.

REDUCTION OF OCCULTATIONS OBSERVED IN 1930.

B. M. Peek, M.A.

The following are the particulars of the reduction of eight occultations of stars by the Moon observed in 1930; they are given in two tables for convenience in printing.

In the last column of Table I the number refers to the page of the *Nautical Almanac* for 1930 from which the star's place was taken; the places of the stars referred to T.W. were kindly furnished by Mr. T. Whitwell.

Occultation No. 1 was observed at Mr. C. S. Saxton's observatory at Whatley, Frome, Somerset, the remainder with the 12¼-inch reflector at Solihull; the co-ordinates of both observatories are given in the *Nautical Almanac* for 1932.

A reduction of No. 1 was made with reference to the place given on p. 456 of the *N.A.* before it was realised that a better place was to be found on p. 284. The computation was repeated for the more reliable star place, and both results are given in the table.

All the phenomena recorded are disappearances, of which No. 6 only was observed at the bright limb. My usual method of timing by means of a stop-watch registering tenths of a second was employed, and as in previous years the practice has been followed of correcting for personal lag by times which average about $0^{s}\cdot5$.

The method of reduction employed is that given by Innes,* 0·182 minute having been added to the recorded time before taking the Moon's place and parallax from the tables, this being the correction recommended by Professor E. W. Brown for the year 1930.

TABLE I

No.	Date 1930	G.M.T. h m s	Star	Mag.	R.A. to Date h m s	Dec. to Date ° ′ ″	Authority
1 (*a*)	Jan. 8	21 47 48·6	*o* Piscium	4·5	1 41 41·23	+ 8 48 22·7	284
(*b*)	...	...	...	...	1 41 41·27	+ 8 48 22·0	456
2	Mar. 10	21 55 12·2	ω Cancri	6·1	7 56 42·95	+25 35 14·7	457
3	Apr. 6	22 34 41·1	*c* Geminorum	5·5	7 39 51·38	+25 57 14·9	457
4	June 6	22 59 56·8	*h* Virginis	5·4	13 29 18·19	− 9 48 29·7	458
5	Nov. 1	17 51 51·0	ψ^1 Aquarii	4·5	23 12 16·56	− 9 27 54·0	459
6	6	21 53 49·9	δ Arietis	4·5	3 7 40·90	+19 28 8·5	294
7	Dec. 26	19 41 48·7	B.D. −7°6027	8·4	23 26 2·05	− 7 13 5·6	T.W.
8	26	21 2 43·7	B.D. −7°6036	6·4	23 27 27·04	− 6 40 11·7	T.W.

TABLE II

No.	$\sigma' - \sigma$	$\chi - \rho$	Sin $(\chi - \rho)$	Cos $(\chi - \rho)$	Remarks
	″	° ′			
1 (*a*)	−2·15	22 3·4	+·3755	·9268	Observed at Whatley.
(*b*)	−1·61	22 6·1	+·3763	·9265	
2	−0·95	330 14·0	−·4965	·8681	
3	+0·45	27 0·1	+·4540	·8910	
4	+1·00	62 32·2	+·8873	·4612	
5	−0·18	326 30·2	−·5519	·8339	
6†	−1·73	331 18·8	−·4800	·8773	
7	−0·29	31 44·2	+·5260	·8505	
8	−0·63	332 3·7	−·4685	·8835	

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE NIZAMIAH OBSERVATORY, HYDERABAD, DURING THE YEARS 1930 AND 1931.

(Communicated by the Director)

Ref. No.	Date		Star	Mag.	Phase	G.M.T. h m s	Observer
1	1930 Jan.	3	κ Capricorni	4·8	D	13 46 38·3	M
2		5	ψ^1 Aquarii	4·5	D	14 48 51·2	M
3	Feb.	9	136 Tauri	4·6	D	13 54 59·7	B
4		9	136 Tauri	4·6	R	15 19 22·1	B
5		12	90 H^1 Cancri	6·1	D	14 23 17·7	V
6		12	90 H^1 Cancri	6·1	R	15 23 36·0	B

* *Union Observatory Circular*, No. 56, 1922 Sept. 12.

† Bright limb. Sudden disappearance not seen, but actual contact observed and watch started about 1 second later. Time error should not exceed 1 second.

Ref. No.	Date	Star	Mag.	Phase	G.M.T.	Observer
					h m s	
7	Mar. 4	54 Ceti	6·0	D	13 53 34·4	M
8	9	49 Aurigæ	5·1	D	13 17 15·5	V
9	18	*i* Libræ	4·3	D	18 59 29·3	M
10	18	*i* Libræ	4·3	R	20 13 11·0	M
11	18	25 Libræ	6·0	R	20 19 33·0	M
12	21	10 G Sagittarii	5·7	R	21 21 17·5	M
13	Apr. 7	ν^1 Cancri	5·7	D	17 18 31·9	M
14	7	ν^2 Cancri	6·4	D	18 2 6·9	M
15	May 5	90 H^1 Cancri	6·0	D	15 36 33·5	M
16	6	η Leonis	3·5	D	13 15 2·6	M
17	Oct. 17	42 Leonis	6·1	R	22 46 8·6	M
18	Dec. 31	ζ Arietis	4·8	D	20 41 33·4	M
19	1931 Jan. 31	54 Aurigæ	5·8	D	13 37 22·5	V
20	Feb. 11	α Scorpii	1·3	D	2 5 18·2	B
21	11	α Scorpii	1·3	R	3 20 2·9	B
22	12	62 B Sagittarii	6·0	R	23 1 56·7	M
23	21	171 B Piscium	6·3	D	14 40 50·8	M
24	26	47 B Aurigæ	6·0	D	14 39 58·5	B
25	Mar. 11	210 B Scorpii	5·8	R	21 7 38·5	M
26	Apr. 24	*v* Geminorum	4·3	D	13 18 22·3	M

Instrument used :

15-inch Grubb refractor, except in the following cases : No. 24 was observed with the 10-inch following telescope of the Astrograph, and Nos. 22 and 23 with a 3-inch portable telescope.

Observers :

B. = T. P. Bhaskaran.
M. = M. K. Bappu.
V. = M. V. Vaidynathan.

Notes

13, 14, 15 : seen through clouds. 6, 21 : time somewhat uncertain.

Nizamiah Observatory,
Hyderabad (Deccan),
India.

OCCULTATIONS OF STARS BY THE MOON, OBSERVED AT THE ASTRONOMICAL OBSERVATORY, KIEW, IN THE YEARS 1929, 1930 AND 1931.

S. D. Tscherny.

(Communicated by the Secretaries)

No.	Date	G.M.T.	Star	Mag.	Dis- or Re-ap.	Tele-scope and Power	Notes
	1929	h m s					
1	Apr. 18	19 16 23·90	η Leonis	3·6	D	R145	Satisfactory.
2	18	19 53 26·32	η Leonis	3·6	R	R145	Satisfactory.
3	June 15	21 5 29·10	κ Virginis	5·7	D	R145	...
4	Sept. 15	21 21 15·28	35 Capricorni	6·0	D	R145	Instantaneous.
5	18	19 40 42·86	376 B Aquarii	6·3	D	R145	Satisfactory.
6	19	19 41 41·94	14 Ceti	5·4	D	R145	Satisfactory.
	1930						
7	Apr. 3	17 5 14·17	B.D. +23° 683	8·5	D	R145	Satisfactory.
8	3	17 5 16·66	B.D. +24° 654	7·3	D	R145	Satisfactory.
9	3	17 5 57·55	62 Tauri	6·1	D	R145	Good.
10	June 2	19 30 52·69	107 B Leonis	6·3	D	R145	Clouds.
11	Aug. 5	20 17 55·74	62 B Sagittarii	6·0	D	R145	Good.
	1931						
12	Jan. 23	17 0 27·94	24 Piscium	6·1	D	R145	Good.
13	Apr. 22	20 2 56·56	406 B Tauri	5·6	D	R145	Very good.
14	June 21	19 29 35·97	χ Leonis	4·6	D	R145	Instantaneous.
15	23	20 22 34·76	319 B Virginis	6·3	D	R145	Good.

Mean Places of Occulted Stars B.D. +23° 683 *and B.D.* +24° 654

	α 1930·0	δ 1930·0	Authority
	h m s	° ′ ″	
B.D. +23° 683	4 19 44·29	+24 8 32·2	Berlin B 1416 (8·3)
B.D. +24° 654	4 19 16·06	+24 14 39·4	Berlin B 1406 (7·3)

Instrument used :

R = 243 mm. Merz-Repsold refractor.

Astronomical Observatory, Kiew :
1931 *December* 24.

THE DEVELOPMENT OF THE DISTURBING FUNCTION WITH LARGE VALUES OF THE RATIO OF THE DISTANCES.

Ernest W. Brown, D.Sc., F.R.S.

1. In an earlier paper * I have shown that, by the device of analytic continuations, we can obtain more rapid convergence of the series which represents any coefficient in the harmonic development of the disturbing function. The series are initially given as expanded in powers of $p = \alpha^2/(1-\alpha^2)$, where α is the ratio of the mean distances of the planets from the Sun. They are then successively expanded in powers of $p-1$, $p-2$, $p-3$, $p-4$, and it appears that a feature which was before noticeable in the expansions along powers of $p-\frac{1}{2}$, $p-1$, namely, more rapid convergence of the coefficients, becomes even more marked in the further continuations.

The series in powers of $p-1$, given earlier to eight places, enables us to calculate the coefficients to this degree of accuracy up to $\alpha = 0{\cdot}82$. But there are certain problems, notably the theory of the mutual perturbations of Titan and Hyperion, satellites of Saturn, which require still higher values. The series given at the end of this paper permit of calculation to eight places up to $\alpha = 0{\cdot}91$ and to four or five places up to $\alpha = 0{\cdot}93$. As a matter of fact, these series, together with those for $p-1$, will serve for all actual problems with present-day needs.

2. This feature of rapid convergence, being equivalent to rapid diminution of successive differences for tabulated values with p as argument, decided Dr. Brouwer and myself to calculate and publish new tables for the development of the disturbing function. These tables are nearing completion, and it may be of some interest to mention their scope and form.

The functions expanded in Fourier series with argument S are

$$\Delta^{-1}, \qquad \Delta^{-3}, \qquad \Delta^{-5}, \qquad \Delta^{-7},$$

where

$$\Delta^2 = 1 - 2\alpha \cos S + \alpha^2.$$

A well-known formula † gives

$$\frac{1}{(1-2\alpha\cos S+\alpha^2)^s} = \frac{\frac{1}{2}C_s^{(0)}g_s^{(0)}}{(1-\alpha^2)^s} + \sum_i \frac{\alpha^i}{(1-\alpha^2)^s} C_s^{(i)} g_s^{(i)} \cos iS \qquad (i = 1, 2, \cdots)$$

where

$$C_s^{(i)} = 2 \cdot \frac{s(s+1) \ . \ . \ . \ (s+i-1)}{i\,!}, \qquad (i \neq 0,\ C_s^{(0)} = 2),$$

$$g_s^{(i)} = 1 - \frac{s \cdot (1-s)}{1 \cdot (i+1)} p + \frac{s(s+1) \cdot (1-s)(2-s)}{1 \cdot 2 \cdot (i+1)(i+2)} p^2 - \cdots. \qquad (2.1)$$

* *Monthly Notices*, **88**, 460, 461, 1928.

† Tisserand, *Mécanique Céleste*, **1**, 279.

The logarithms of $C_s^{(i)} \cdot g_s^{(i)}$ will be given to eight places of decimals from $i=0$ to $i=11$, at intervals of p of ·01 from $p=0$ to $p=2{\cdot}5$, with first differences, for $s=\frac{1}{2}, \frac{3}{2}, \frac{5}{2}$; and to seven places for $s=\frac{7}{2}$.

For higher values of p, the series for each value of i in powers of $p-3$, $p-4$ will be given; the first five coefficients of $(1-\alpha^2)\Delta^{-1}$, $(1-\alpha^2)^2\Delta^{-3}$ will also be tabulated at intervals of 0·001 from $\alpha=0{\cdot}900$ to $\alpha=0{\cdot}950$ to six significant figures. Certain additional numerical results, useful for expansions by harmonic analysis, will also be printed.

3. The method used for obtaining the series given below will be described briefly. It was originally started as an experiment to find out how a straightforward calculation of a series with a great many terms could be best performed. As the work was done under conditions which precluded the use of computing machines, the formulæ were arranged for mental calculation. It turned out to be easy and rapid, and numerous checks were available. It was started *ab initio* with more places of decimals than had been used in my former paper, and the results were, in fact, used as the basis of the tables mentioned in the previous paragraph.

Since $s=\frac{1}{2}$ in this case, the symbol G_i is adopted instead of the more complicated $g_s^{(i)}$. The following formulæ were used.

From a known property of the hypergeometric series, it may be shown that G_i satisfies the differential equation,

$$(p+p^2)\frac{d^2G_i}{dp^2}+(2p+1+i)\frac{dG_i}{dp}+\tfrac{1}{4}G_i=0. \tag{3.1}$$

This may be integrated in the form

$$(p+p^2)\frac{dG_i}{dp}+iG_i+\tfrac{1}{4}\int G_i dp=\text{const.} \tag{3.2}$$

The result of differentiating (3.1) $n-1$ times is

$$(p+p^2)\frac{d^{n+1}G_i}{dp^{n+1}}+(2np+n+i)\frac{d^nG_i}{dp^n}+(n-\tfrac{1}{2})^2\frac{d^{n-1}G_i}{dp^{n-1}}=0. \tag{3.3}$$

If G_i be expanded in powers of $p-\kappa$ by Taylor's theorem, we have

$$G_i=\kappa_0-\kappa_1(p-\kappa)+\kappa_2(p-\kappa)^2 \ . \ . \ . \ , \quad \kappa_n=\frac{(-1)^n}{n!}\left(\frac{d^nG_i}{dp^n}\right)_{p=\kappa}. \tag{3.4}$$

On substituting this in (3.3) and equating to zero the coefficient of $(p-\kappa)^{n-1}$, we obtain a relation between three successive coefficients which may be written

$$\kappa_{n+1}=\frac{1}{(\kappa+\kappa^2)(n+1)}\left\{(2\kappa n+n+i)\kappa_n-\left(n-1+\frac{1}{4n}\right)\kappa_{n-1}\right\}. \tag{3.5}$$

From (3.5), knowing κ_0, κ_1, that is, the values of G_i, dG_i/dp for $p=\kappa$, we can obtain the remaining coefficients by recurrence formulæ. The rapid convergence for large values of κ is evidently due mainly to the divisor $\kappa+\kappa^2$.

This gives the coefficients in (3.4) for a given value of i. To obtain the

corresponding expansions for lower values of i, we calculate them successively from the formula

$$G_{i-1} = G_i + (p - \kappa + \kappa)\frac{1}{i}\frac{dG_i}{dp}, \tag{3.6}$$

which is deduced easily from the series (2.1). This formula illustrates a second advantage of starting with the highest value of i needed, and working down, the first evidently being in the use of (2.1) with large values of i.

4. The following steps show the method of procedure :—

Step 1.—Calculation of G_{10}, $\int_0^1 G_{10}dp$, dG_{10}/dp, d^2G_{10}/dp^2 for $p = 1$ from the series (2.1) with $s = \frac{1}{2}$, $i = 10$, the third and fourth of these functions being used as checks with their calculation by (3.2), (3.1). If the coefficient of $(-p)^j$ in (2.1) be denoted by a_j, we have

$$a_{j+1} = \frac{a_j}{11+j}\left(j + \tfrac{1}{4} \cdot \frac{1}{j+1}\right),$$

by means of which the successive coefficients were calculated, starting with $a_0 = 1$. The work was carried out in four groups, the last term of each group being also calculated directly from the formula. The last group closing with the coefficient of $-p^{31}$ was obtained in the form

$$a_{23}p^{22}(p - a_2'p^2 + \ldots - a_9'p^9).$$

The successive differences

$$1, \qquad 1 - a_2', \qquad 1 - 2a_2' + a_3', \qquad \ldots$$

are the coefficients of the series in brackets when it is changed to a series in powers of $p/(p+1)$ or of α^2. This converged so rapidly with $p = 1$, or $\alpha^2 = \frac{1}{2}$, that the result for G_{10} with this value of p, carried out to fourteen places of decimals, could be relied on to the thirteenth place.

The calculation of the other three functions was easily performed after the coefficients in G_{10} had been found.

Step 2.—The results of step 1 give κ_0, κ_1 for $p = 1$, $i = 10$. The remaining κ_n were then calculated successively from (3.5) with $\kappa = 1$. The last sensible coefficient is correct to one or two units in the tenth place, a loss of accuracy of some three to four places occurring in the continued use of the formula. The result gives G_{10} in powers of $p - 1$.

Step 3.—Calculation in powers of $p - 2$. From (3.2) we have, with the limits 2, 0 and with $i = 10$,

$$\left(\frac{dG_{10}}{dp}\right)_{p=2} + \tfrac{10}{6}(G_{10} - 1)_{p=2} + \tfrac{1}{24}\int_0^2 G_{10}dp = 0.$$

The series for G_{10} in powers of $p - 1$ was used to calculate the second and third terms, and gave the value of the first term with little loss. Thence, by recurrence from (3.5) with $\kappa = 2$, we obtain the remaining coefficients in the series for G_{10} in powers of $p - 2$.

Step 4.—Calculation of the coefficients in the series for G_9, G_8, . . . G_0, in powers of $p - 2$. These were obtained rapidly from (3.6) with $\kappa = 2$. There is practically no loss of accuracy in the process.

Step 5.—Calculation in powers of $p-3$. For some reason which I do not now recall (the work was finished over a year ago), the start was made with $i=9$ instead of with $i=10$. From (3.2) with the limits 3, 1 and with $i=9$, we obtain

$$12\left(\frac{dG_9}{dp}\right)_{p=3} - 6\left(\frac{dG_9}{dp}\right)_{p=1} + 9(G_9)_{p=3} - 9(G_9)_{p=1} + \tfrac{1}{4}\int_1^3 G_9 dp = 0.$$

For the calculation of all but the first term we use the expression of G_9 in powers of $p-2$; the result furnishes the first term, and thence, with the help of (3.5), the remaining coefficients of the series in powers of $p-3$. The calculation of G_8, G_7, . . . G_0 is performed as before.

Step 6.—Calculation in powers of $p-4$. This was deduced from the series in powers of $p-3$ in the same manner as the latter was deduced from the series in powers of $p-2$.

5. The results for G_9, G_8, . . . G_0 in powers of $p-4$ are given below. For testing purposes the sum of each series for $p=3$ is also given. The formula is

$$G_i = c_0 + c_1(p-4) + c_2(p-4)^2 + \ldots + c_n(p-4)^n + \cdots,$$

and the table contains the values of the c_n for each value of i. The maximum error revealed by the tests is two units in the ninth place, the calculations being carried to ten places.

n	G_9	G_8	G_7	G_6	G_5
0	+·925502596	+·919082360	+·911375330	+·901927269	+·890028938
1	−· 14445531	−· 15414059	−· 16534106	−· 17847496	−· 19414392
2	+· 716098	+· 806710	+· 917547	+· 1055766	+· 1232356
3	−· 51390	−· 60561	−· 72297	−· 87666	−· 108383
4	+· 4477	+· 5487	+· 6832	+· 8668	+· 11261
5	−· 441	−· 560	−· 724	−· 956	−· 1298
6	+· 47	+· 62	+· 83	+· 114	+· 161
7	−· 5	−· 7	−· 10	−· 14	−· 21
8	+· 1	+· 1	+· 1	+· 2	+· 3
$p=3$	+·940720586	+·935369807	+·928906930	+·920927951	+·910796813

n	G_4	G_3	G_2	G_1	G_0
0	+·874497425	+·853171925	+·821536568	+·767789218	+·642637681
1	−· 21325500	−· 23726517	−· 26873675	−· 31287884	−· 38389461
2	+· 1465179	+· 1785631	+· 2255657	+· 3023290	+· 4621143
3	−· 137379	−· 180098	−· 248004	−· 370727	−· 666541
4	+· 15079	+· 21036	+· 31160	+· 51023	+· 104615
5	−· 1824	−· 2689	−· 4246	−· 7525	−· 17249
6	+· 236	+· 366	+· 611	+· 1163	+· 2938
7	−· 32	−· 52	−· 92	−· 186	−· 512
8	+· 5	+· 8	+· 14	+· 30	+· 91
9	−· 1	−· 1	−· 2	−· 5	−· 16
10				+· 1	+· 4
11					−· 1
$p=3$	+·897442660	+·878888323	+·850950029	+·802531052	+·686440252

Yale University :
1931 *December* 1.